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Letter

The Hopf equation with certain modular nonlinearities

Efim Pelinovsky [a,c], Tatiana Talipova [a,b], Ekaterina Didenkulova [b,c,*]

[a] Il'ichev Pacific Oceanological Institute Far Eastern Branch Russian Academy of Sciences, 43 Baltiyskaya Str., 690041 Vladivostok, Russia
[b] A.V. Gaponov-Grekhov Institute of Applied Physics of the Russian Academy of Sciences, 46 Ulyanov Str., 603950 Nizhny Novgorod, Russia
[c] Faculty of Informatics, Mathematics and Computer Science, HSE University, 25/12 Bolshaya Pecherskaya Str., 603155 Nizhny Novgorod, Russia

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ABSTRACT

The dynamics of waves of sign-variable shape has been studied within the framework of the Hopf equation ($u_t + Fu_x = 0$) with a non-analytic propagation velocity containing the modulus of the function at the zero crossing ($F \sim |u|^\alpha$). It is shown that Riemann waves exist only for a certain smoothness of the function $F[u(x)]$ at the initial moment of time. Otherwise, the wave immediately overturns (gradient catastrophe). Popular in nonlinear physics the modular Hopf equation, the dispersionless Schamel equation, and the dispersionless logarithmic Korteweg-de Vries equation are considered as examples.

Introduction

The Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation, as well as the Gardner equation that unites them, now form the standard equations of nonlinear wave physics and appear in many branches of physics as a first approximation for weakly nonlinear and weakly dispersive waves using asymptotic methods, which use Taylor series for analytical functions describing nonlinearity. Then, various versions of the generalized Korteweg-de Vries equation appeared:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

with nonlinear function $f(u)$, which represents a general form, including its non-analyticity. In a number of ocean applications (internal waves, Rossby waves), the nonlinear function is represented as a high-degree polynomial [1–4]. Although such equations are no longer integrable from the point of view of nonlinear physics, many properties of solutions to Eq. (1) with polynomial nonlinearity have been well studied. In particular, the solitons in the framework of such equations can have a very complex form such as pyramidal solitons [5,6]. It is also easy to obtain criteria for the modulation instability of wave packets [7,8]. However, it should be taken into account that if a nonlinear function contains a positive term of the type u^5 or higher (highest degree), then the solutions to Eq. (1) "explode", and the nature of these solutions has been studied in detail in a number of works; see, for example, [9–11]. From the mathematical point of view, Eq. (1) with polynomial nonlinearity is well-posed, which allows the use of the entire arsenal of rigorous mathematical methods.

Meanwhile, in a number of applications, non-analytic functions appear in Eq. (1) that describe nonlinearity. A striking example is the Schamel equation derived in 1973, when in Eq. (1) the nonlinear function was represented by the following expression $f(u) \sim |u|^{3/2}$ [12]. Originally obtained in plasma physics for ion-acoustic waves, in recent years, the Schamel equation has been actively used to describe waves in metamaterials [13–15] and electrical circuits [16,17]. Although the Schamel equation is not integrable, like the KdV and mKdV equations, within its framework, there are solitons of both polarities that are resistant to interaction between themselves and external forces in the sense that inelastic interaction leads to a small radiation [18–20]. However, in the case of a large ensemble of solitons, the statistical characteristics of the soliton gas do not become stationary as is the case of integrable systems, so that radiation plays a decisive role in the nonstationarity of the wave field [21].

In the Fermi-Pasta-Ulam chains, under a certain law of particle interaction, the so-called logarithmic Korteweg-de Vries equation appeared [22–25]. In this equation, the nonlinear function is $f(u) \sim u\log(|$

* Corresponding author at: Faculty of Informatics, Mathematics and Computer Science, HSE University, 25/12 Bolshaya Pecherskaya Str., 603155 Nizhny Novgorod, Russia.

E-mail address: edidenkulova@hse.ru (E. Didenkulova).

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$u|$). Logarithmic generalizations have also been obtained for other popular nonlinear evolution equations (Zakharov-Kuznetsov equation, Benjamin–Bona–Mahony (BBM) equation, etc.); see, for example, [26]. In this and similar equations, the solitons have the form of Gaussian pulses, but their nonstationary dynamics have not yet been studied.

Subsequently, the number of equations of the KdV type with a non-analytic nonlinear function only increased. In particular, the modular equation with $f(u) \sim |u|$ appeared in the bimodular theory of elasticity [27,28], and then the "canonical" modular equation with $f(u) \sim u|u|$ [27,29], which differs from the canonical KdV equation only in the modulus of the wave function. Within the framework of the equation with $f(u) \sim |u|$, there are no solitons on the zero pedestal as all unipolar solutions are linear. Within the framework of the canonical modular KdV equation, it is easy to construct families of multisoliton solutions of the same polarity because in this case we are dealing with the classical KdV equation. New effects arise here during the interaction of solitons of different polarities, when large pulses can be formed, while the radiation is also small. It is important to note the series of works concerning the modular Burgers equation (in acoustics, dissipation often prevails over dispersion) [30–37].

Finally, a class of "sublinear" equations when $f(u) \sim u|u|^b$ with $b < 0$ appeared in dynamics of elasticity [38,39]. In this case, the solitons with exponential tails are impossible, and solitary waves have the form of compactons limited in space. The compactons are stable formations as the radiation during interaction is small, but their dynamics are much more complex than the dynamics of exponential or algebraic solitons [38,39].

Of course, it is possible to "unite" various nonlinear functions in practical problems, and in the literature, one can find equations with names like the Schamel-KdV equation or the Schamel-logarithmic KdV equation, etc. Sometimes such equations are referred to as the generalized Gardner equation. The presence of non-analytic nonlinear functions makes it very difficult to prove the existence and uniqueness theorems mathematically with only a couple of publications on this topic [37,39].

In this work, we consider the dispersionless limit of- Eq. (1) in the case of certain nonlinear nonanalytic functions. The resulting equation can be called the Hopf equation by analogy with the case of quadratic nonlinearity. To the best of our knowledge, only the case of the Hopf equation with nonlinearity as $u|u|$ has been studied. In particular, the spectrum of a Riemann wave was obtained initially having the form of a monochromatic wave [27,40]. Other published papers do not cover the analysis of wave breaking in the Hopf equation with non-analytic nonlinear function. Our methodology is based on the method of characteristics for unidirectional wave propagation. As it will be shown, within the framework of the modular Hopf equation, it is not always possible to consider the classical formulation of the problem of the evolution of a sine wave. This makes it possible to evaluate the role of nonlinear effects for waves crossing the zero level (where the non-analyticity of the nonlinear function is manifested).

Wave breaking

The Hopf equation with an arbitrary nonlinear function (propagation speed) $F(u)$ has the following form

$$\frac{\partial u}{\partial t} + F(u)\frac{\partial u}{\partial x} = 0. \tag{2}$$

The solution to Eq. (2) determines the Riemann wave

$$u = \Psi[\zeta],\ \ \zeta = x - F(u)t, \tag{3}$$

where $\Psi(x)$ determines the initial waveform. A graphical representation of the Riemann wave is made below. First, we examine the conditions for the breaking of a Riemann wave (gradient catastrophe). To do this, it is enough to calculate the steepness of the wave (its gradient)

$$\frac{\partial u}{\partial x}(x,t) = \frac{\Psi'}{1+\Psi'\frac{dF}{du}t},\ \ \Psi' = \frac{d\Psi}{d\zeta}. \tag{4}$$

Wave breaking (crossing of characteristics) begins at the moment of time when the denominator in (4) turns to zero, i.e.

$$T = max\left(\frac{-1}{\frac{d\Psi}{d\zeta}\frac{dF}{du}}\right) = max\left(\frac{-1}{\frac{dF(\Psi)}{dx}}\right), \tag{5}$$

and at large times, solution (3) ceases to function. It follows that to calculate the breaking time, it is enough to analyze the spatial distribution of the field at the initial moment of time, or rather the speed of propagation $F(x)$.

The zone of low field values, where the nonlinear function $F(u)$ is non-analytic, is of greatest interest. Therefore, let us present it in the form

$$F(u) = q|u|^{\alpha}, \tag{6}$$

where for definiteness $q > 0$ and α is an arbitrary number. We work with the initial disturbances with norm L^2, that is, square integrable on an infinite interval or with periodic functions. Therefore, although $F(x) \geq 0$, its derivative dF/dx changes sign within the interval or at its end, so that the breaking always occurs on one of the two slopes of the wave (either frontal or rear).

Let us consider a typical formulation of the problem, when at the initial moment of time, a monochromatic wave with amplitude A and wave number k is given:

$$u(x,0) = \Psi(x) = A\sin kx. \tag{7}$$

If we use formula (5), then it is easy to show that wave breaking occurs at no more than two points (we consider only one period) on the initial profile in the intervals $kx \in [\pi/2,\ \pi]$ and $kx \in [3\pi/2,\ 2\pi]$ (in the classical case of the Hopf equation with quadratic nonlinearity, wave breaking begins at only one point $kx = \pi$) during the following time

$$T = \frac{1}{qk\alpha A^{\alpha}} min\left(\frac{1}{|\sin kx|^{\alpha-1}|\cos kx|}\right). \tag{8}$$

When $\alpha \geq 1$, a simple formula follows:

$$T = \frac{1}{qk\alpha A^{\alpha}}\frac{\alpha^{\alpha/2}}{(\alpha-1)^{(\alpha-1)/2}} \tag{9}$$

and discontinuity forms at the points of the second and fourth quadrant

$$\sin^2(kX) = \frac{\alpha-1}{\alpha} \tag{10}$$

In particular, in the case of the modular Hopf equation, the breaking time is [40]

$$T = \frac{1}{qkA}. \tag{11}$$

and the breaking occurs on the initial profile at points $kX = \pi$ and $kX = 2\pi$. Recall that in the case of the dispersionless classical Korteweg-de Vries equation, the breaking occurs at only one point $kX = \pi$.

In the case of the dispersionless modified Korteweg-de Vries equation (cubic nonlinearity), this also implies the well-known result [41]

$$T = \frac{1}{qkA^2}. \tag{12}$$

and the breaking occurs at points $kX = 3\pi/4$ and $kX = 3\pi/2$.

In fact, the non-analyticity of the propagation speed $F(u)$ in case of $\alpha \geq 1$ does not appear, since the derivative dF/dx exists for all final values of the argument. A different situation occurs when $\alpha < 1$. In this

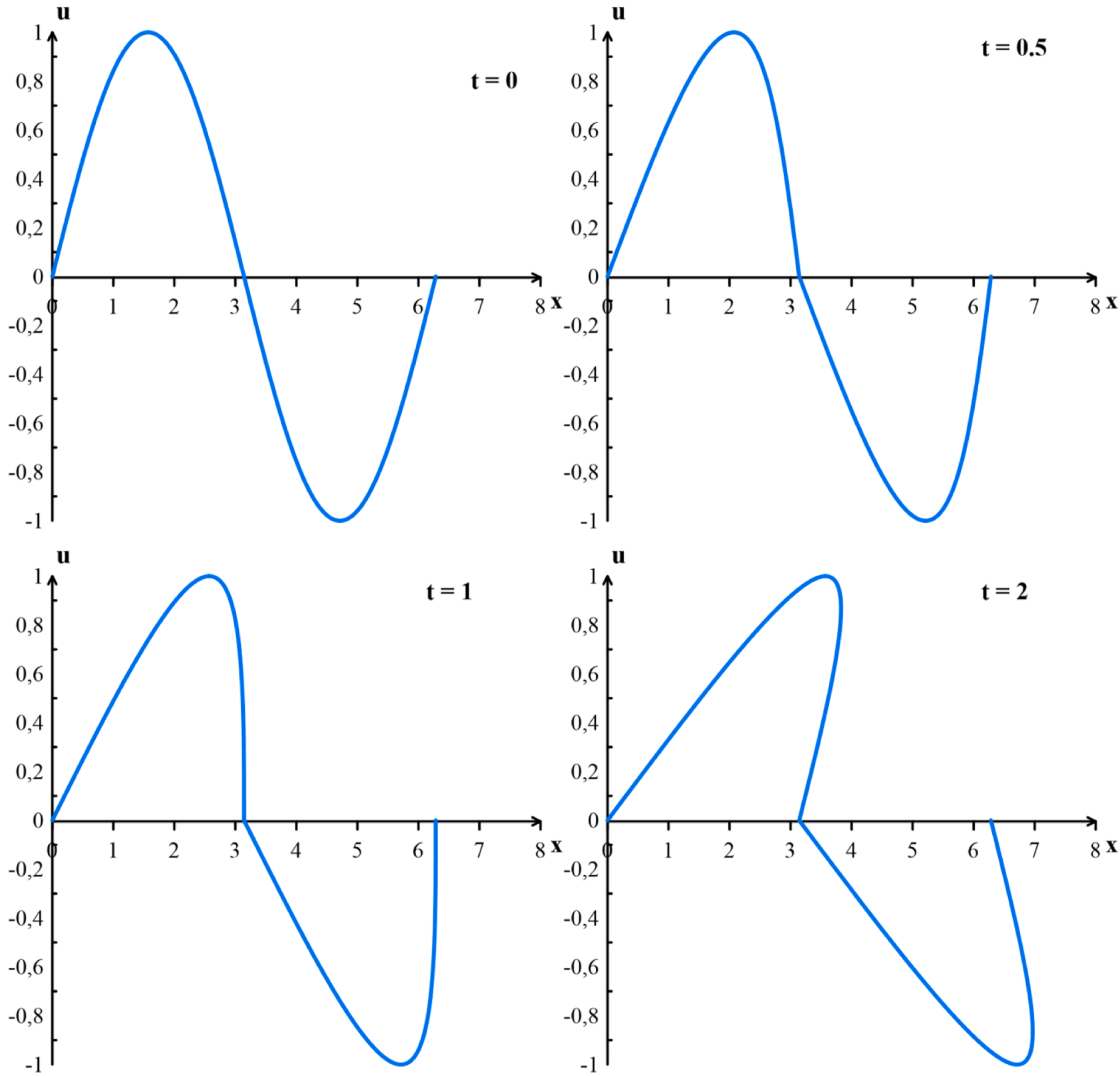

Fig. 1. Transformation of an initially sine wave ($\beta = 1$) within the framework of the modular Hopf equation, $T = 1$.

case, the breaking time is always zero, and the wave immediately breaks at the points where $u = 0$. The explanation for this is quite simple: nonlinear speed *F(x)* in the case of $\alpha < 1$ has a steep section with infinity derivative already at the initial moment, and for this function, the gradient catastrophe occurs immediately. Thus, the finiteness of the wave steepness *u(x)* at the initial moment of time does not automatically lead to the existence of a smooth solution (at least on a finite time interval); this requires the finiteness of the propagation velocity gradient *F (x)*. In essence, this follows from the transformation of Eq. (2) to the canonical form of the Hopf equation [42]

$$\frac{\partial F}{\partial t} + F\frac{\partial F}{\partial x} = 0, \tag{13}$$

which is possible if the derivative *dF/du* exists and is not equal to zero. In the case of a non-analytic nonlinear function *F(u),* derivative *dF/du* does not exist at zero, and the calculation of derivatives has to be done more carefully considering the limits of the corresponding expressions.

Thus, given that $\alpha < 1$, the initially sine wave immediately overturns at the points $u = 0$. In relation to the well-known Schamel equation α =1/2, this means that a dispersionless limit does not exist for any initial perturbations with a non-zero steepness (gradient), and the dispersion in it is manifested at all times. Obviously, the same situation should be realized when the interaction of two solitary waves of different polarity is considered. In the zone of their intersection, the zero level always intersects with a non-zero derivative. This, in our opinion, explains the relatively strong radiation during the interaction of solitons of different polarities within the framework of the Schamel equation, which influenced the statistical characteristics of the soliton gas [21].

Below, a number of examples of solving Eq. (2) with different initial conditions is considered. In this case, we do not limit ourselves only to the zone of existence of the Riemann wave, but also extend the solutions to longer times when the wave becomes multi-valued. As in similar problems of dispersion hydrodynamics within the framework of KdV-type polynomial equations [43,44], they allow us to better understand the places of formation of shock fronts, soliton groups, and undular bores on a long-wave body.

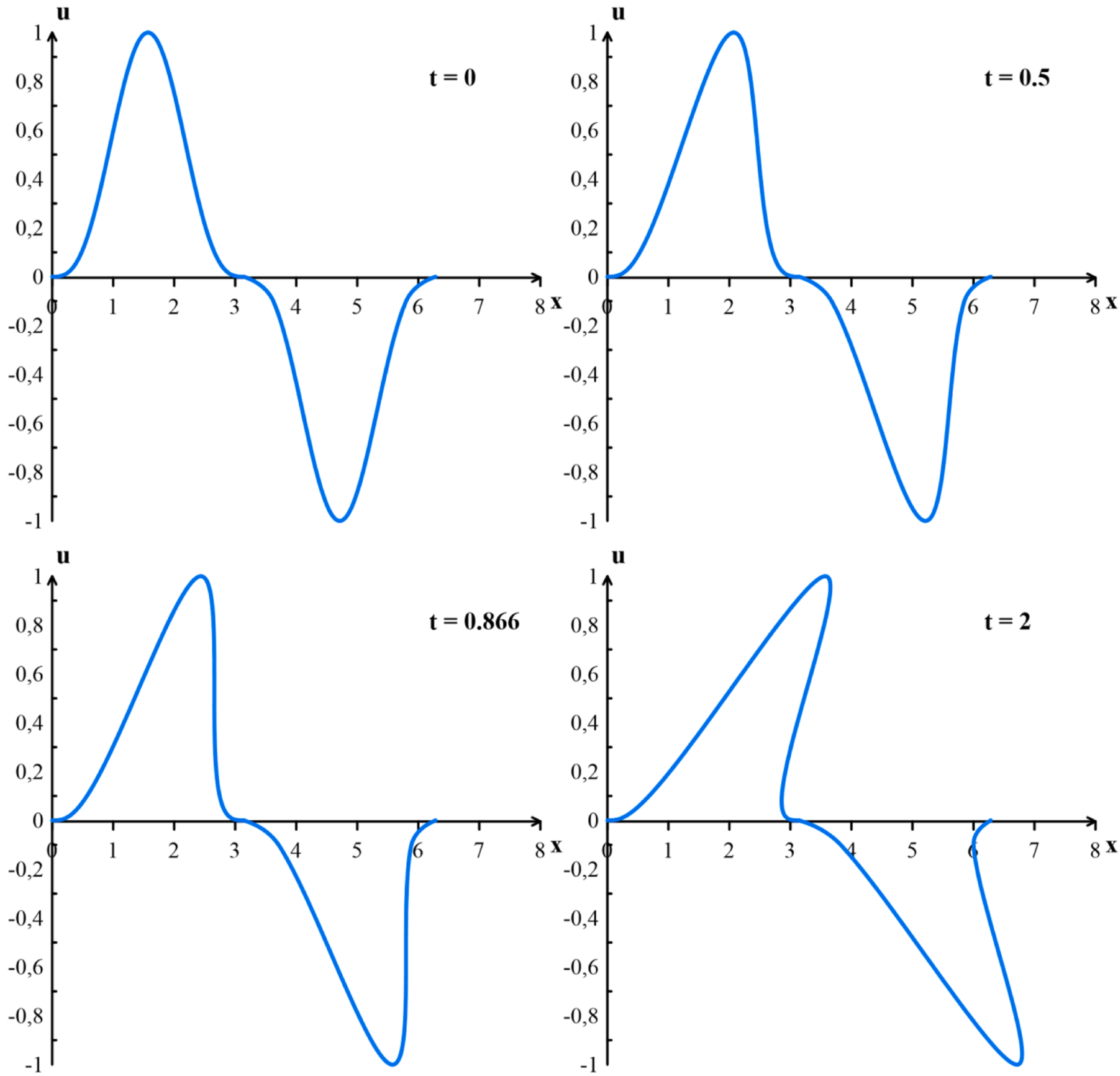

Fig. 2. Wave transformation ($\beta = 3$) within the framework of the modular Hopf equation. The moment of gradient catastrophe is $T = 0.866$.

Nonlinear wave dynamics within the framework of the modular Hopf equation

Let us consider in more detail the evolution of a periodic wave within the framework of the modular Hopf equation [27,40]

$$\frac{\partial u}{\partial t} + |u|\frac{\partial u}{\partial x} = 0 \tag{14}$$

with initial periodic disturbance

$$u(x,0) = \Psi(x) = \left|sin^{\beta-1}(x)\right|\sin x \tag{15}$$

with $\beta \geq 1$. All constants in the equation and initial condition can be easily removed by appropriate scaling. The Riemann wave solution has the following form

$$u(x,t) = \left|\sin^{\beta-1}(x-|u|t)\right|\sin(x-|u|t). \tag{16}$$

It exists for a finite time $t < T$, where T time of the breaking (time of gradient catastrophe) is

$$T = \frac{\beta^{\frac{\beta}{2}-1}}{(\beta-1)^{\frac{\beta-1}{2}}}, \tag{17}$$

and discontinuity forms on the front slope at points

$$\sin^2 X = \frac{\beta-1}{\beta}. \tag{18}$$

If a purely monochromatic wave is given ($\beta = 1$), the time of breaking is $T = 1$, and the breaking (overlap) begins at points $X = \pi$ and $X = 2\ \pi$ as we discussed above. If the wave has the shape (16) with $\beta > 1$, then the overlap begins at a higher point in a shorter time. In particular, when $\beta = 3$, the time of breaking is $T = \sqrt{3}/2 \approx 0.866$, and the breaking occurs at the points $X = \pi - arcsin\sqrt{\frac{1}{3}} \approx 2.6$ and $X = 2\pi - arcsin\sqrt{\frac{1}{3}} \approx 5.7$. At the stage of a Riemann wave, it is possible to find its Fourier spectrum analytically [27,40]. It should be noticed that even at the stage of the Riemann wave, the solution is not smooth and its derivative becomes discontinuous when passing the level $u = 0$. In essence, the upper and

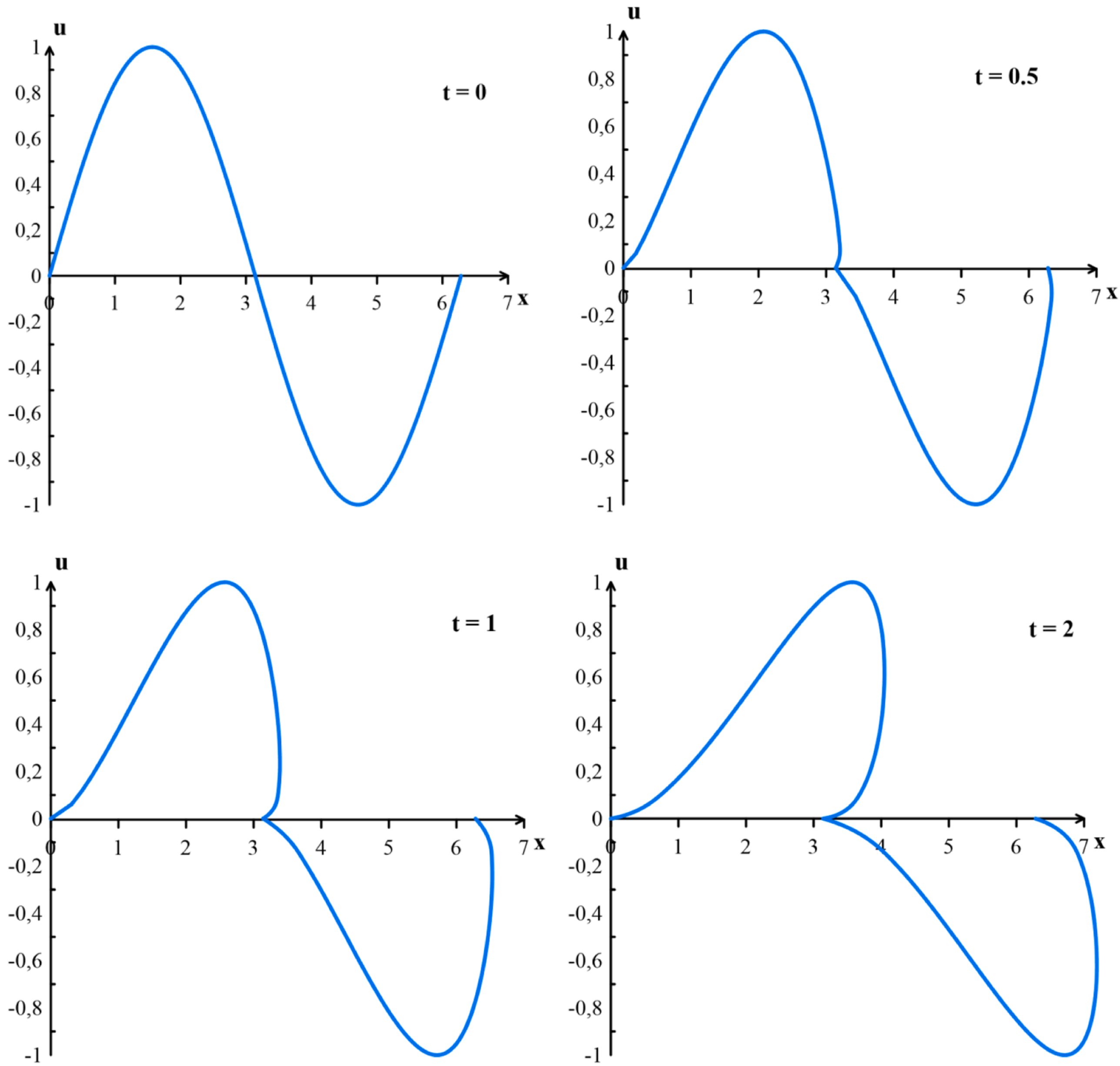

Fig. 3. Transformation of an initially sine wave ($\beta = 1$) within the framework of the Schamel equation ($T = 0$).

lower halves of the wave are deformed independently, steepening their leading front.

Nonlinear deformation of an initially sine wave ($\beta = 1$) on the interval $[0,\ 2\pi]$ is shown in Fig. 1. The given pictures confirm what was said above: the upper and lower halves of the wave are transformed equally with the steepening of the leading front, and the gradient catastrophe occurs at the points $u = 0$, as follows from the theory. Then, the wave overlaps, and the top of the wave, as well as its bottom, overtake points with a small value of the variable u. The back slope of the wave tends to an self-similar profile over time x/t. However, the derivative jumps near the level $u = 0$ saved at any time.

In the case of a smoother form of the initial disturbance ($\beta = 3$), gradient catastrophe occurs at a higher level $u = 0.44$ for a shorter time (T =0,866) as follows from (17). This process is illustrated in Fig. 2. At level $u = 0$, the solution is now smoother.

Nonlinear wave evolution within the framework of the Schamel equation

Let us consider in more detail the evolution of a periodic wave within the framework of the dispersionless Schamel equation

$$\frac{\partial u}{\partial t} + \sqrt{|u|}\ \frac{\partial u}{\partial x} = 0 \tag{19}$$

with an initial periodic disturbance of the form (16). As already mentioned, in this equation, the initially sine wave immediately acquires a steep front at the level of $u = 0$, which then spreads to the top and bottom of the wave. Formally, there are other points of overlap on the wave body, such as those shown in Fig. 2 for the modular Hopf equation. However, the time of their formation is long and they are not visible against the background of a multivalued wave (Fig. 3).

Due to the symmetry of the propagation speed, the positive and negative half-waves are transformed in the same way, breaking at the leading front. The back slope of the wave becomes smoother and it is described by a unique Riemann solution at all times. The back slope is

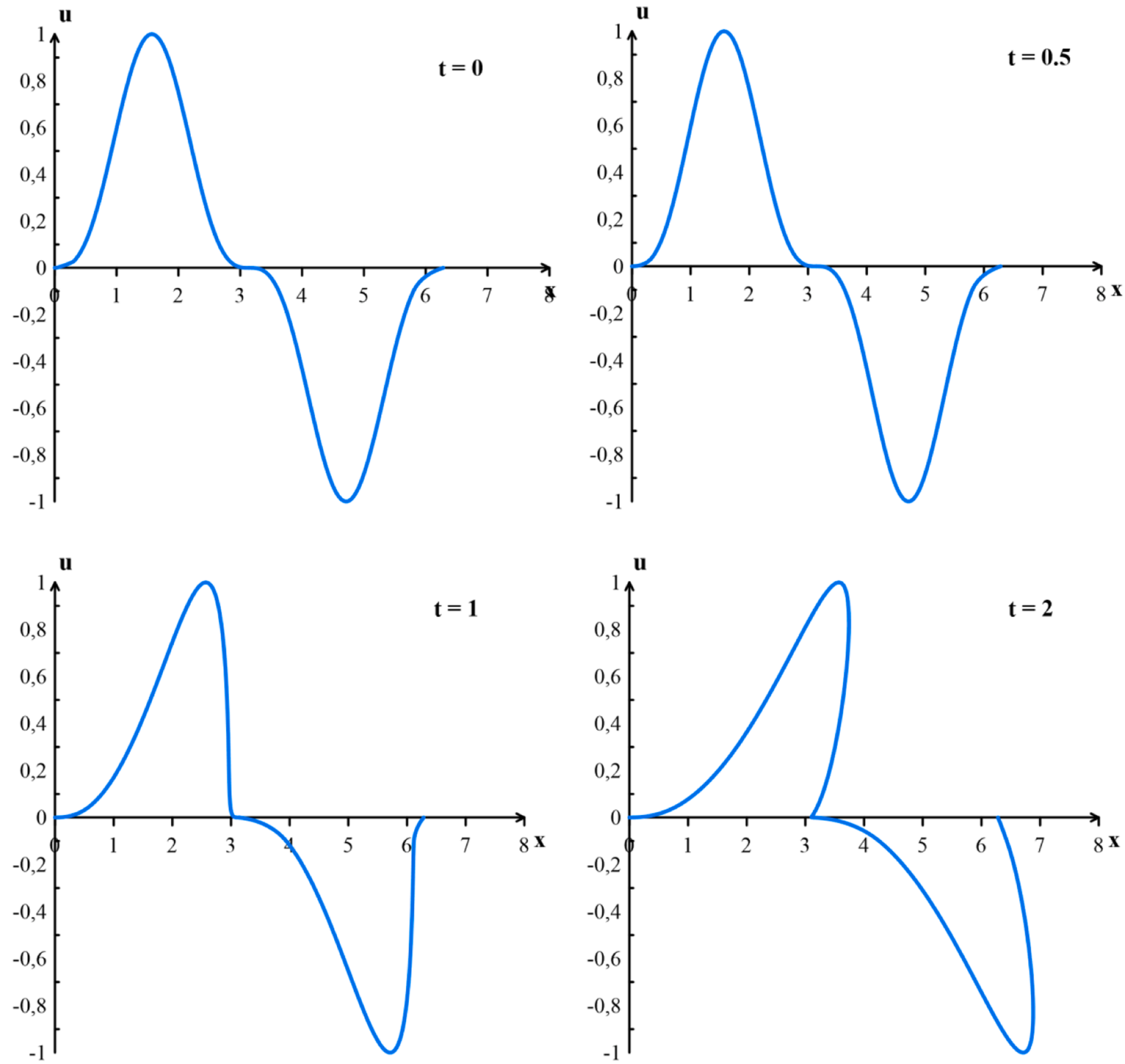

Fig. 4. Wave transformation of the type (17) with $\beta = 3$ within the framework of the Schamel equation.

described by an asymptotically self-similar solution to Eq. (16) - a "rarefaction wave" of parabolic shape which, for a positive half-wave, has the form (at small x) $u \sim (x/t)^2$.

If we consider smoother initial disturbances, for example a wave (16) with $\beta = 3$, then the speed profile $F(x)$ at the initial moment of time has a finite gradient. Therefore, first, there is a transformation of the wave as a Riemann wave (Fig. 4) and then, after the breaking time ($T = \sqrt{2} / \sqrt[4]{3} \approx 1.075$), the solution becomes multi-valued. The discontinuity is formed at the point $|u| = 3^{-3/2} = 0.19$, and then spreads along the body of the wave. Over time, there is also a jump in the derivative at the level $u = 0$.

The Hopf equation with logarithmic nonlinearity

Special consideration is required for the Hopf equation with logarithmic nonlinearity, which arises when dispersion is neglected in the logarithmic KdV equation and some other similar evolution equations:

$$\frac{\partial u}{\partial t} + \ln(|u|)\frac{\partial u}{\partial x} = 0. \tag{20}$$

Although formally the solution to this equation in the form of a Riemann wave has the form

$$u(x,t) = \Psi[x - \ln|u|t], \tag{21}$$

however, the difficulties in choosing the initial function $u(x)$ are immediately visible when the wave does not break immediately. If we consider periodic disturbances with zero average (of the type chosen in the previous sections), then the transition through the axis $u = 0$ results in an infinite gradient in propagation speed $V(u) = \ln(|u|)$, and, consequently, to the immediate wave breaking. This process is illustrated in Fig. 5 in the case of an initial disturbance in the form of a monochromatic wave (7). The top and bottom of the wave $|u| = 1$ (points with $x = \pi/4$ and $x = 3\pi/4$) stands still, and all other points shift over time to the left in accordance with their amplitude and velocity gradient du/dx. Discontinuity at points $x = \pi$ and 2π is formed instantly as these points move to the left at infinite speed. The "tails" of the resulting curves do

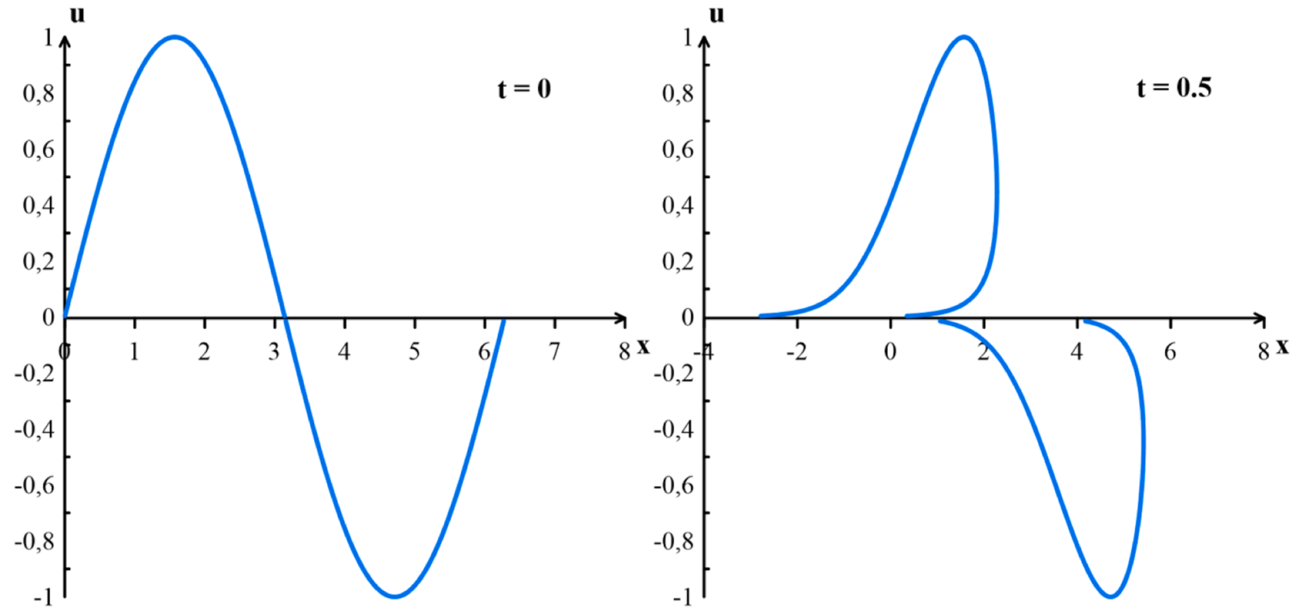

Fig. 5. Transformation of a wave of the form (7) within the framework of the logarithmic Hopf equation.

not merge; here there is a change in sign of du/dx. As a result, a steep front still forms on the right slope of each "half" of the wave.

Final gradient in the initial velocity profile $V(x)$ is achieved only in the case of constant sign initial perturbations, but then, in essence, the non-analyticity of the Hopf equation does not manifest. We will not analyze such disturbances. The main conclusion from the analysis of the logarithmic Hopf equation is that it is the most "nonlinear" equation in this class, which does not allow existence of the Riemann waves of sign-variable shape.

Conclusion

The dynamics of waves of sign-variable shapes has been studied within the framework of the Hopf equation ($u_t + Fu_x = 0$) with a non-analytic propagation velocity containing the modulus of the function at the zero crossing ($F \sim |u|^{\alpha}$). Such equation is appeared in various physical contexts (ion-sound waves in plasma, waves in electrical circuits, waves in metamaterials, the chains with Herzian potential, etc.). It is shown that Riemann waves exist only for a certain smoothness of the function $F[u(x)]$ at the initial moment of time, otherwise the wave immediately breaks (gradient catastrophe). Popular in nonlinear physics the modular Hopf equation, the dispersionless Schamel equation, and the dispersionless logarithmic Korteweg-de Vries equation are considered as examples. In the framework of the modular Hopf equation, a sine wave is transformed into a Riemann wave and exists for a finite time. In the framework of the Schamel equation, initially, a sine wave breaks immediately, and Riemann waves can be the disturbances that pass the zero level as $x|x|$ or with higher powers. Within the framework of the logarithmic Korteweg-de Vries equation, a sign-variable disturbance of any shape breaks immediately.

Further dynamics of the wave near the breaking point and further depends on the dispersion. In integrable systems, the solitons appear on a long-wave body which, under certain conditions, again converge into a sine wave (a well-known recurrence phenomenon). For modular equations of the Korteweg-de Vries type, such a problem has not yet been studied.

CRediT authorship contribution statement

Efim Pelinovsky: Writing – original draft, Methodology, Investigation, Conceptualization. **Tatiana Talipova:** Writing – review & editing, Methodology, Investigation, Conceptualization. **Ekaterina Didenkulova:** Writing – review & editing, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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