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A generalization of the split-step Padé method to the case of coupled acoustic modes equation in a 3D waveguide

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ABSTRACT

The split-step Padé approach is an extremely efficient tool for the integration of pseudodifferential parabolic equations, which are widely used for the modeling of acoustic wave propagation. In this study, a generalization of this method to the case of parabolic equations with unknown vector functions is presented. Such generalization requires an algorithm for efficient numerical evaluation of a function of a matrix with differential operators as its elements. After a finite-difference discretization this algorithm reduces to the solution of several Sylvester-like problems. The generalized split-step Padé method presented here is an attractive tool for solving 3D problems of sound propagation in the ocean within the framework of coupled mode parabolic equations theory.

1. Introduction

Wide-angle parabolic equations are currently widely used for numerical modeling of wave propagation in underwater acoustics [1,2] (as well as in some other fields of physics, e.g., [3,4]). Arguably, the most advanced tool for their solution is the so-called split-step Padé method, proposed independently by Collins [1] and Avilov [5]. The method consists in a formal factorization of the operator in the scalar Helmholtz equation of the form

$$u_{xx} + u_{yy} + k^2 u = \left(\partial_x + \mathrm{i}\sqrt{k^2 + \partial_y^2}\right)\left(\partial_x - \mathrm{i}\sqrt{k^2 + \partial_y^2}\right) u = 0\,, \tag{1}$$

where $u = u(x, y)$ is an unknown function (e.g., acoustic pressure), $k^2 = k^2(x, y)$ is the wavenumber or the refractive index (subscripts x, y denote partial derivatives with respect to these variables throughout this study). A one-way counterpart of (1), corresponding to sound propagation in the positive direction of the x-axis is called a *pseudodifferential parabolic equation* (PDPE) [5,6]

$$\left(\partial_x - \mathrm{i}\sqrt{k^2 + \partial_y^2}\right) u = 0\,, \tag{2}$$

since it contains a pseudodifferential operator $\mathrm{i}\sqrt{k^2 + \partial_y^2}$.

The *split-step Padé* (SSP) method [1,6] of solving Eq. (2) consists in the formal integration of this equation over the interval $[x, x + h]$, which leads to the equality

$$u(x + h, y) = \exp\left(\mathrm{i}h\sqrt{k^2 + \partial_y^2}\right) u(x, y) \equiv \hat{\mathcal{P}} u(x, y)\,, \tag{3}$$

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and in a replacement of the operator $\hat{\mathcal{P}}$ on the right-hand side of this equality (usually called the *propagator*) by its $[p/p]$ Padé approximant of the form

$$\exp\left(\mathrm{i}h\sqrt{k^2+\partial_y^2}\right)u \approx \mathrm{e}^{\mathrm{i}k_0 h}\left(d_0+\sum_{k=1}^{p}\frac{d_k}{1+b_k\hat{X}}\right)u = \mathrm{e}^{\mathrm{i}k_0 h}\left(d_0 u+\sum_{k=1}^{p}d_k w_k\right), \tag{4}$$

where k_0 is a reference value of $k(x,y)$, and $\hat{X}=(k^2-k_0^2+\partial_y^2)/k_0^2$. The quantities $w_k = w_k(x,y)$ are obtained by solving the following operator equations

$$\left(1+b_k\hat{X}\right)w_k = u,$$

which are transformed into e.g. linear systems with tridiagonal matrices by the standard second-order finite difference discretization of the operator $\hat{X}$ [1].

Thus, for a $[p/p]$ Padé approximation of the square root exponential operator, the marching numerical scheme requires p inversions of some sparse (e.g., tridiagonal) matrices for each step along the x-axis.

In the present study, this method is generalized to the case of a coupled system of equations of the form (1). Such systems arise naturally when considering the propagation of coupled normal modes in a 3D geoacoustic waveguide [2,7]. Although the corresponding one-way equations have been considered in the literature (see, e.g., [2]), the algorithm proposed below is novel. It could serve as a basis for new full-wave 3D models of sound propagation in the ocean, offering much higher computational efficiency than the 3D parabolic equations that are currently widely used in underwater acoustics [8]).

2. The coupled system of elliptic equations and its one-way counterpart

The main goal of this study is to generalize the SSP algorithm outlined in the previous section to the case of a coupled system of J elliptic equations [2,9]

$$u_{m,xx}+u_{m,yy}+k_m^2u_m+\sum_{j=1}^{J}V_{jm}u_j+\sum_{j=1}^{J}W_{jm}u_{j,y}=0, \tag{5}$$

where $\mathbf{V}(y)=(V_{jm}(y))$ and $\mathbf{W}(y)=(W_{jm}(y))$ are y-dependent $J\times J$ matrices (typically full). In fact, the Eqs. (5) can also contain the terms with $u_{j,x}$, but without any loss of generality they can be eliminated by a change of variables.

In particular, such equations naturally emerge from the decomposition

$$U(x,y,z)=\sum_{m=1}^{M}u_m(x,y)\phi_m(z,x,y), \tag{6}$$

of the acoustic field $U(x,y,z)$ in a shallow sea over vertical normal modes $\phi_m(z)$ (which also depend parametrically on x,y) as described in [2,9]. The functions u_m are called *mode amplitudes*.

Let us first introduce a vector-matrix form of the coupled Eqs. (5), which is essential for further discussion. First, we define a vector function $\mathbf{u}(x,y)=(u_1,u_2,\dots,u_J)$ consisting of the unknown functions (we emphasize that it is a row vector). The system of scalar Eqs. (5) can now be rewritten in vectorized form as

$$\mathbf{u}_{xx}+\mathbf{u}_{yy}+\mathbf{u}\mathbf{K}^2+\mathbf{u}_y\mathbf{W}+\mathbf{u}\mathbf{V}=0, \tag{7}$$

where $\mathbf{W}(x,y)$, $\mathbf{V}(x,y)$ are the coupling matrices, and $\mathbf{K}=\mathrm{diag}(k_1,k_2,\dots,k_M)$ is a diagonal matrix with the quantities $k_m(x,y)$ along the main diagonal, i.e., such that $\mathbf{K}_{mn}=\delta_{mn}k_m$).

Introducing the operator $\hat{R}_M$ of right multiplication by any matrix $\mathbf{M}$, we rewrite Eq. (7) as

$$\mathbf{u}_{xx}=-\partial_y^2\mathbf{u}-\hat{R}_W\partial_y\mathbf{u}-\hat{R}_V\mathbf{u}. \tag{8}$$

Next, we can perform the formal factorization of (8) and obtain the following one-way equation along the x direction

$$\mathbf{u}_x=\mathrm{i}\sqrt{\partial_y^2+\hat{R}_W\partial_y+\hat{R}_V}\,\mathbf{u},\quad x>0. \tag{9}$$

This equation contains the square root of a matrix with differential operators as elements. In principle, it is a pseudodifferential parabolic equation for an unknown vector function. In the following it will be called *vectorized PDPE* (VPDPE).

We can formally integrate (9) over the interval $[x,x+h]$ to get

$$\mathbf{u}(x+h,y)=\exp\left(\mathrm{i}k_0h\sqrt{1+\hat{X}}\right)\mathbf{u}(x,y),$$

where k_0 is a reference value of k_j (it can be the same for all j), and the operator $\hat{X}$ is defined by the formula

$$\partial_y^2+R_Q\partial_y+R_S=k_0^2(1+\hat{X}).$$

3. Generalized split-step Padé approach for solving a VPDPE

As in the adiabatic (uncoupled) case, the propagator allows an approximation by an exponential of the form

$$\mathbf{u}(x+h,y) = \mathrm{e}^{\mathrm{i}k_0 h}\left(d_0 + \sum_{k=1}^{p} \frac{d_k}{1+c_k\hat{X}}\right)\mathbf{u}(x,y) = \mathrm{e}^{\mathrm{i}k_0 h}\left(d_0\mathbf{u}(x,y) + \sum_{k=1}^{p} d_k\mathbf{f}_k\right). \tag{10}$$

The terms $\mathbf{f}_k(x,y)$ in this case are obtained by solving the following operator equations

$$(1+c_k\hat{X})\,\mathbf{f}_k(x,y) = \mathbf{u}(x,y)\,. \tag{11}$$

In the following, we consider the solution of an Eq. (11) for one step of the marching scheme separately, i.e., we fix the value of x and omit the subscript k for the moment

$$(1+c\hat{X})\,\mathbf{f} = \mathbf{u}.$$

To perform a finite-difference discretization of Eq. (11) in y, we replace the vector function $\mathbf{u}(y)$ by a matrix $\mathbf{U} = (U_{\ell j})$ with N rows (number of grid points in y) and J columns (number of modes). Then the matrices $\mathbf{f}_k(y) \sim \mathbf{F}_k(y)$ can be found by solving the following matrix equation (where we drop the subscript as explained above)

$$k_0^2\left(\frac{1}{c}-1\right)\mathbf{F} + \mathbf{D}_2\mathbf{F} + \hat{\mathcal{L}}(\mathbf{F}) + \mathbf{F}\mathbf{S} = \mathbf{B}_n\,, \tag{12}$$

where the linear operator $\hat{\mathcal{L}}$ acting on $N\times J$ matrices is defined as

$$(\hat{\mathcal{L}}(\mathbf{F}))_{k\ell} = \sum_{n=1}^{N}\sum_{j=1}^{J}(\mathbf{D}_1)_{kn}\,\mathbf{F}_{nj}\,\mathbf{Q}_{j\ell n}\,, \tag{13}$$

$\mathbf{D}_2$ and $\mathbf{D}_1$ are square $N\times N$ matrices corresponding to finite-difference approximations of the second and the first derivatives with respect to the transverse variable y, respectively, and, by definition, $\mathbf{F}_{nj} = (\mathbf{f})_j(y_n)$.

Note that the p matrix Eqs. (12) to be solved at each step of the marching scheme are similar to the so-called *generalized Sylvester problem*. The only difference is that after the finite difference discretization in y, the functions $\mathbf{Q}(y)$ and $\mathbf{S}(y)$ become tensors $J\times J\times N$. Despite this difference, the Eqs. (11) can be solved by exactly the same vectorization procedure and construction similar to the tensor product of matrices. Let us replace the matrix $\mathbf{F}$ by a column vector $\mathrm{vec}(\mathbf{F})$ which is obtained by stacking columns of the matrix $\mathbf{F}$ (this is called 'vectorizing' a matrix). Then the operator $\hat{\mathcal{L}}$ can be replaced by multiplication by a block matrix $\mathrm{vec}(\hat{\mathcal{L}})$ of size $NJ\times NJ$ defined as

$$\mathrm{vec}(\hat{\mathcal{L}}(\mathbf{F})) = \mathrm{vec}(\hat{\mathcal{L}})\ \mathrm{vec}(\mathbf{F}) = \left[\begin{array}{c|c|c} \bar{Q}_{11}D_1 & \dots & \bar{Q}_{1J}D_1 \\ \hline \vdots & \ddots & \vdots \\ \hline \bar{Q}_{J1}D_1 & \dots & \bar{Q}_{JJ}D_1 \end{array}\right]\mathrm{vec}(\mathbf{F})\,, \tag{14}$$

where

$$\bar{Q}_{j\ell} = \mathrm{diag}(Q_{j\ell 1},\dots,Q_{j\ell N}) = \begin{bmatrix} Q_{j\ell 1} & & \\ & \ddots & \\ & & Q_{j\ell N}\end{bmatrix}$$

is a diagonal matrix $N\times N$ consisting of the values of the function $Q_{j\ell}(y)$ at the grid points in y.

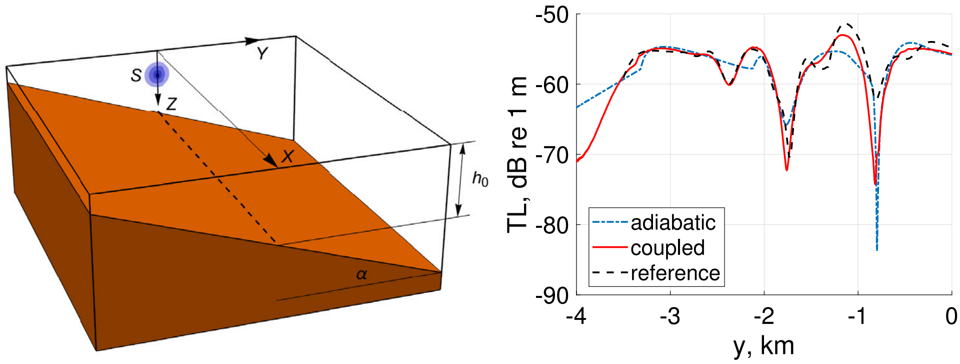
Note that the computations reported below were performed using MATLAB routines for constructing the Kronecker product of matrices and for matrix inversion. The code was surprisingly efficient despite the fact that we had $J = 30$ and $N = 4000$, which is already suitable for handling practical products. In the benchmark case described below, the computation time in MATLAB was about one hour (for the range $0 < x < 25$ km, -3.5 km $< y < 3.5$ km), which is much faster than the state-of-the-art implementation of the 3D PE [8] in C++.

4. An example: propagation in the wedge

In this section, we consider the solution of an equation of the form (7) that naturally arises in shallow water acoustics. Consider a typical area of shallow water near the coast, shown in Fig. 1(a) with a time-harmonic point source of the frequency $f = 25$ Hz placed at the point S. We do not give an exact description of the test case parameters, as they are provided in numerous papers using this benchmark (see, e.g., [6–8,10]).

In this case, mode amplitudes $u_m(x,y)$ satisfy the Eqs. (5), where $k_m = k_m(y)$ is the horizontal wavenumber of the mth mode (the media in this case parameters do not depend on x), and V_{jm}, W_{jm} are coupling coefficients, which can be easily found after computing ϕ_j (the computation of ϕ_j is done by solving a Sturm–Liouville problem, and numerous methods are available to do this in a very efficient way, see [9]).

Although sufficiently far from the source the field can be calculated by the adiabatic counterpart of (5) (i.e. the uncoupled system where all V_{jm}, W_{jm} are set to zero and we have an equation of the type (2) for each mode amplitude), at close range, where upslope/downslope propagation plays an important role, this simplification does not work. Similarly, simpler parabolic equations with mode coupling (see, e.g., [7]) cannot be used in this case due to the limitations of their aperture in the horizontal plane.



(a) (b)

Fig. 1. (a) The coastal wedge: an acoustic waveguide where mode amplitudes are governed by the system (5). (b) Acoustic field in the wedge along the line $z = 30$ m, $x = 1000$ m as a function of y, computed by solving the VPDPE (9) using the generalized SSP algorithm (red line) and by its adiabatic counterpart (2) for each mode (blue line). The reference solution is represented by the black dashed line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In order to achieve high accuracy of the solution in this case, we set $p = 11$ and $J = 30$. Our goal is to compute the acoustic field along the line $z = 30\,\mathrm{m}$, $x = 1000\,\mathrm{m}$ and to compare the results with the benchmark solution obtained by the source image method. Note that we used starters (the Cauchy data modeling the point source) proposed in [6] and truncated the computational domain by using the same PML as in [6].

The plot of the acoustic field $P(x, y, z)$ (in dB re $P(1\,\mathrm{m})$) obtained by solving VPDPE (9) using the SSP method is shown in Fig. 1(b). It can be clearly seen that our solution (red solid line) exhibits a very good agreement with the reference (black dashed line). It can be observed that the adiabatic solution obtained using the Eqs. (2) for each mode amplitude is very inaccurate in the considered region (although it can be used in the far field as shown in [6]).

5. Conclusion

In this study, a generalization of the SSP algorithm to the case of a one-way wave equation for an unknown vector function is presented. This algorithm can be used to model coupled mode propagation in 3D waveguides in underwater acoustics. It can be considered as a direct generalization of the method [6] to the non-adiabatic propagation scenarios. On the one hand, the coupled mode parabolic wave Eq. (9) is a generalization of its scalar counterpart, while on the other hand it can be considered as a generalization of the vectorized WKBJ method to the case of 3D problems. The proposed method allows full-wave 3D modeling of sound propagation with much lower computational cost than 3D parabolic equations (which are currently widely used for this purpose).

CRediT authorship contribution statement

Pavel S. Petrov: Conceptualization, Formal analysis, Investigation, Methodology, Software, Supervision, Validation, Writing – original draft. **Matthias Ehrhardt:** Investigation, Methodology, Supervision, Writing – review & editing. **Sergey B. Kozitskiy:** Investigation, Software, Validation, Visualization, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] M.D. Collins, A split-step Padé solution for the parabolic equation method, J. Acoust. Soc. Amer. 93 (1993) 1736–1742.

[2] A.T. Abawi, W.A. Kuperman, M.D. Collins, The coupled mode parabolic equation, J. Acoust. Soc. Am. 102 (1) (1997) 233–238.

[3] M.S. Lytaev, Numerov-Padé scheme for the one-way Helmholtz equation in tropospheric radio-wave propagation, IEEE Anten. Wirel. Prop. Lett. 19 (12) (2020) 2167–2171.

[4] Y.Y. Lu, Some techniques for computing wave propagation in optical waveguides, Commun. Comput. Phys. 1 (6) (2006) 1056–1075.

[5] V. Avilov, Pseudodifferential parabolic equations of sound propagation in the slowly range-dependent ocean and its numerical solution, Acoust. Phys. 41 (1995) 1–7.

[6] P.S. Petrov, M. Ehrhardt, A.G. Tyshchenko, P.N. Petrov, Wide-angle mode parabolic equations for the modelling of horizontal refraction in underwater acoustics and their numerical solution on unbounded domains, J. Sound Vibr. 484 (2020) 115526.

[7] M.Y. Trofimov, S. Kozitskiy, A. Zakharenko, A mode parabolic equation method in the case of the resonant mode interaction, Wave Motion 58 (2015) 42–52.

[8] Y.-T. Lin, T.F. Duda, A.E. Newhall, Three-dimensional sound propagation models using the parabolic-equation approximation and the split-step Fourier method, J. Comput. Acoust. 21 (01) (2013) 1250018.

[9] F.B. Jensen, W.A. Kuperman, M.B. Porter, H. Schmidt, Computational Ocean Acoustics, Springer Science & Business Media, 2011.

[10] F. Sturm, Leading-order cross term correction of three-dimensional parabolic equation models, J. Acoust. Soc. Amer. 139 (1) (2016) 263–270.