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Non-integrable soliton gas: The Schamel equation framework

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ABSTRACT

Soliton gas or soliton turbulence is a subject of intense studies due to its great importance to optics, hydrodynamics, electricity, chemistry, biology and plasma physics. Usually, this term is used for integrable models where solitons interact elastically. However, soliton turbulence can also be a part of non-integrable dynamics, where long-lasting solutions in the form of almost solitons may exist. In the present paper, the complex dynamics of ensembles of solitary waves is studied within the Schamel equation using direct numerical simulations. Some important statistical characteristics (distribution functions, moments, etc.) are calculated numerically for unipolar and bipolar soliton gases. Comparison of results with integrable Korteweg–de Vries (KdV) and modified KdV (mKdV) models are given qualitatively. Our results agree well with the predictions of the KdV equation in the case of unipolar solitons. However, in the bipolar case, we observed a notable departure from the mKdV model, particularly in the behavior of kurtosis. The observed increase in kurtosis signifies the amplification of distribution function tails, which, in turn, corresponds to the presence of high-amplitude waves.

1. Introduction

Solitons are the exact solutions of many equations and have many applications in nonlinear dynamics, including optical fibers, surface and internal waves in the ocean, laboratory and astrophysical plasma, etc. According to the classic definition, they are coherent large amplitude pulses whose shape and speed do not change as they propagate. Such particle-like behavior is explained by the balance between dispersion and nonlinearity, which, on one hand, tends to cause wave spreading, and, on the other hand, leads to its steepening. The great importance of solitons lies in their ability to transfer energy over long distances. Among the physical systems in which solitons play an important role is the formation of abnormally large waves (i.e., rogue waves or freak waves). This problem gives rise to the soliton gas theory, which initially emerged within integrable models such as the nonlinear Schrödinger and KdV equations. Usually, the terms "soliton gas" or "soliton turbulence" mean an ensemble of solitons with random parameters in the integrable systems [1,2]. The property of solitons to interact elastically with each other gives rise to an obvious association with a gas of elastically colliding particles. V. Zakharov [1,2] first introduced the concept of soliton turbulence where the kinetic theory of rarefied solitons was build. Later on, this concept has been extended to the dense soliton gas with frequently interacted solitons [3,4]. Kinetic equations describes the transport of spectral data of the associated scattering problem, but there is no information about phases (polarity) of solitons and wave fields themselves. However, the so-called density of states (DOS) of soliton gases described by the kinetic theory can be used to compute averaged fluxes, conserved quantities, and statistical momenta of the asymptotic stage of the evolution in integrable system [4–6]. Direct numerical simulations of wave ensemble is an alternative approach to study of soliton gas statistics. These results for integrable KdV-models can be found in [7–13]. Similar studies concerning wave packets that preserve their energy are called breathers. Their collective dynamics promotes freak wave formation [14,15]. Soliton and breather turbulence are extensively investigated within the nonlinear Schrödinger equation in the contexts of water waves and nonlinear optics [16–22]. Moreover, the presence of soliton and breather turbulence in ocean waves was confirmed in [23,24].

The problem of soliton turbulence may also be investigated in non-integrable models, allowing the existence of soliton-like impulses, which interact almost elastically. However, the analytical methods used to study the integrable turbulence are not applicable here, and only numerical simulation can be applied. Dutykh and Pelinovsky [25] compared the collective behavior of soliton ensembles within the KdV equation the non-integrable KdV–BBM type models using direct numerical simulations. The closeness in the behavior of the wave fields

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was ascertained, including the fact that the probability distributions remain quasi-invariant during the system evolution for both KdV and KdV–BBM cases. In the present article, we study the soliton turbulence within the Schamel equation, which is not integrable by the inverse scattering transform since a Lax pair does not exist for this equation. It describes the development of a localized, coherent wave structure that propagates in plasma [26–29]. This equation contains a modular nonlinear term with non-integer power, and this stands out strongly on background of traditional equations Korteweg–de Vries hierarchy. The features of two soliton collisions in the framework of the Schamel equation were recently investigated in [30,31]. It was shown that the soliton interaction with the same polarity follows the classical scenario of the modified Korteweg–de Vries soliton with small difference due to non-integrability of the Shamel equation. However, in the case of bipolar soliton interactions the energy accumulation in the largest soliton may contribute to the rogue wave formation in the case of long-term wave dynamics. Also as we show such interaction leads to the new features of statistical properties due to transformation of the soliton energy to the dispersive tails. In the present work we study the dynamics of the soliton gas within the Schamel equation and its statistical properties. In the Section 2 the Schamel equation and the numerical methods. In the Section 3 the collective dynamics of ensembles of solitons with the same polarity is considered. Further, in Section 4 we discover the features or bipolar soliton collision and statistical properties of bipolar soliton gas, with special emphasis on freak wave formation in such wave fields. Conclusion is given at the end of the paper.

2. The schamel equation

In our research, we investigate solitary wave interactions by focusing on the Schamel equation in its canonical form

$$u_t + \sqrt{|u|}u_x + u_{xxx} = 0. \tag{2.1}$$

Within this equation, the variable u represents the wave field at a specific position x and time t. It is worth noting that the Schamel equation is a Hamiltonian equation, meaning it possesses a Hamiltonian function that governs its behavior. The Hamiltonian associated with this equation is defined as follows

$$\mathcal{H} = \int_{-\infty}^{+\infty}\left[-\frac{1}{2}u_x^2 + \frac{4}{15}\mathrm{sign}(u)|u|^{5/2}\right]dx. \tag{2.2}$$

By expressing Eq. (2.1) in Hamiltonian form with respect to the functional $\mathcal{H}$, we can establish a relationship between the wave dynamics and the Hamiltonian. This connection is expressed by the following equation

$$u_t = \frac{\partial}{\partial x}\left[\frac{\delta \mathcal{H}}{\delta u}\right],$$

where the functional derivative of the Hamiltonian with respect to u is given by

$$\frac{\delta \mathcal{H}}{\delta u} = u_{xx} + \frac{2}{3}\mathrm{sign}(u)|u|^{3/2}.$$

One intriguing feature of the Schamel equation is the invariance of its Hamiltonian $\mathcal{H}$ due to the absence of explicit time dependence. This invariance implies that the Hamiltonian remains constant throughout the evolution of the wave system. Furthermore, the Schamel equation (2.1) possesses an additional invariant, known as the Casimir invariant or the mass invariant. This quantity is defined by the following integral

$$M(t) = \int_{-\infty}^{+\infty} u(x,t)dx, \tag{2.3}$$

and it characterizes the mass or the total "amount" of the wave field at any given time t. In addition to the mass invariant, the equation also exhibits a momentum invariant, given by

$$P(t) = \int_{-\infty}^{+\infty} u^2(x,t)dx. \tag{2.4}$$

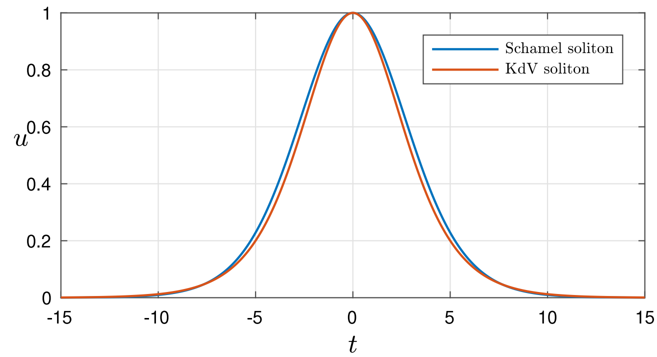

Fig. 1. Soliton profile of the Schamel equation (2.5) and the KdV equation (2.7).

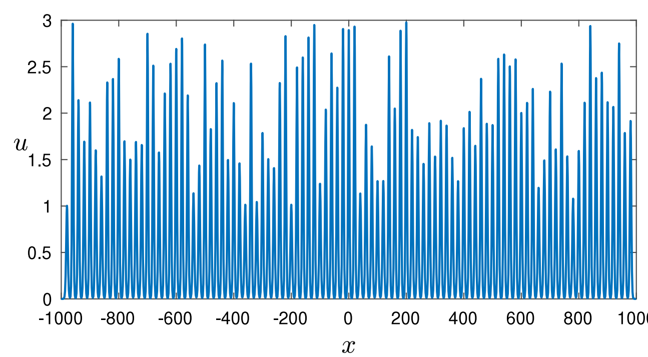

Fig. 2. Initial distribution of solitons.

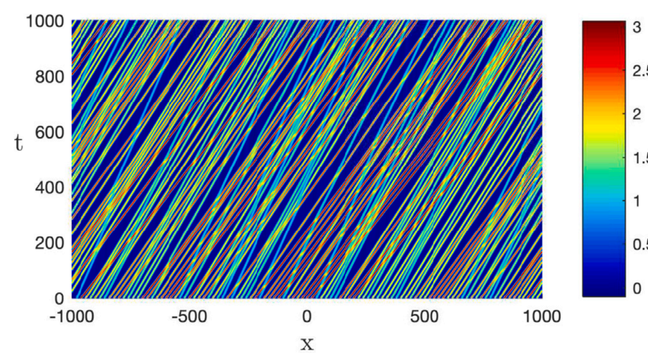

Fig. 3. $x - t$ diagram of soliton field.

These invariants, namely the Hamiltonian (2.2), the mass (2.3), and the momentum (2.4), play a crucial role in evaluating the accuracy and reliability of numerical methods employed to solve the Schamel equation (2.1).

The Schamel equation (2.1) supports solitary wave as solutions. These solitary waves can be described by the following expressions

$$u(x,t) = a\,\mathrm{sech}^4\left(k(x-ct)\right), \text{ where } c = \frac{8\sqrt{|a|}}{15} \text{ and } k = \sqrt{\frac{c}{16}}. \tag{2.5}$$

Here, a stands for the amplitude of the solitary wave, which can be positive or negative. The parameter c denotes the speed of the solitary wave and k characterizes its wavenumber.

The KdV equation

$$u_t + uu_x + u_{xxx} = 0. \tag{2.6}$$

also admits solitary wave as solutions described by the formulas

$$u(x,t) = a\,\mathrm{sech}^2\left(k(x-ct)\right), \text{ where } c = \frac{a}{3} \text{ and } k = \sqrt{\frac{a}{12}}. \tag{2.7}$$

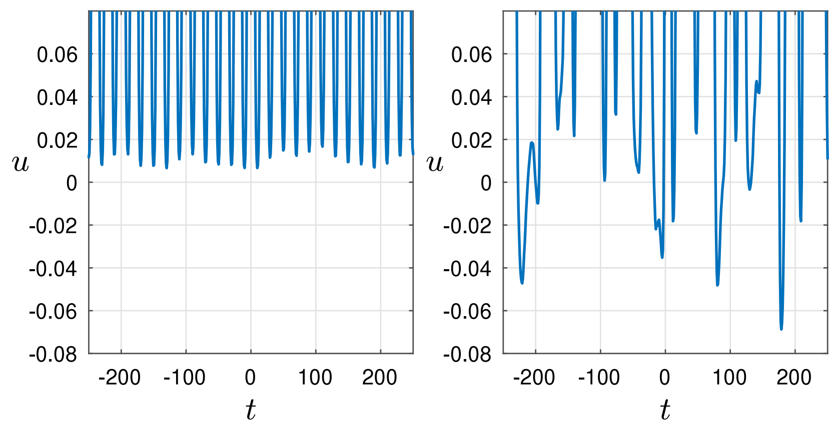

Fig. 4. Zoom on portion of the computational domain at $t = 0$ (left) and at $t = 1000$ (right).

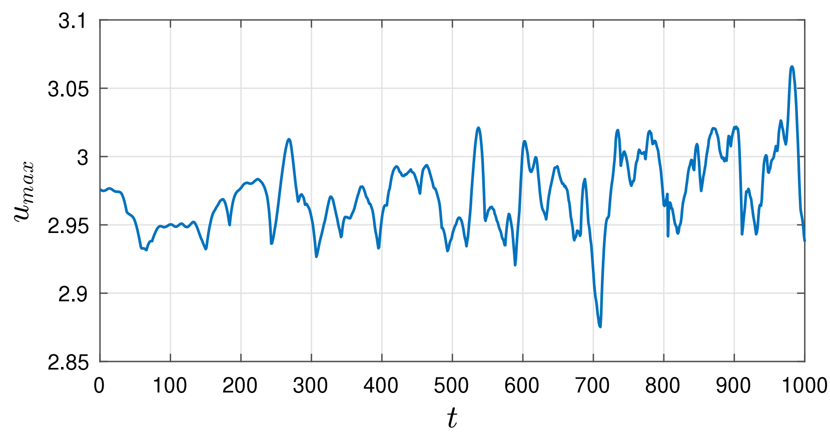

Fig. 5. Temporal variability of the maxima of the wave field.

Soliton solutions of the Schamel equation are wider than the KdV solitons (see Fig. 1), and they propagate faster than KdV ones. Details of soliton interactions for the Schamel equation, such as phase-shift and energy transfer, were recently investigated in the works of Flamarion et al. [30] and Didenkulova et al. [31].

The Schamel equation (2.1) is solved numerically through a Fourier pseudospectral method combined with an integrating factor. The computational domain chosen for the simulation is a periodic interval $[-L, L]$, discretized with an equidistant grid consisting of N points. This grid configuration facilitates precise approximation of spatial derivatives, as discussed in [32]. To mitigate the influence of spatial periodicity, a sufficiently large computational domain is employed. For the temporal evolution of the equation, the classical fourth-order Runge–Kutta method is employed with discrete time steps of size Δt. Typical simulations employ parameter values such as $L = 200$, $N = 2^{13}$, and $\Delta t = 0.005$. Numerical simulations are controlled by retaining of the first and second moments with precision of 10^{-9} and 10^{-8}, respectively.

3. Unipolar soliton

With aim to study the dynamics of unipolar soliton ensemble, we set initial wave field as a sequence of 100 separated solitons with random amplitudes uniformly distributed from the range $[1, 3]$ in random order and fixed distance between their positions is 20 units (see Fig. 2). The solitons propagate to the right and owing to different speeds interact with each other. Non-dimensional time of calculation is set equal to 1000, so that solitons have time to interact. Multiple soliton interactions during the computational time is clearly seen in spatio-temporal diagram (Fig. 3). Due to the repulsion of unipolar solitons, pair soliton interactions predominate here, which were studied in detail in [30]. In the process of wave collision, solitons are phase-shifted, thus after several interactions their locations are hardly predicted.

It is well known that interaction of unipolar KdV-like solitons leads to decrease in amplitude of resulting impulse [30,33–37]. In the non-integrable Schamel equation this property is the same in case of two-soliton collision. However, there is a radiation created by inelastic collisions of solitons which is displayed in details in Fig. 4. A more detailed study on the solitary wave collision for the Schamel equation can be seen in [30]. In this work, the authors investigated how radiation is generated during the solitary wave collisions and the Lax-categorization for solitary wave collisions. While the dispersive tail amplitudes are approximately only 1% of the averaged amplitude of the initial soliton distribution, they still exert a minor influence, leading to a slight increase in the maximum wave field following interactions with other solitons (see Fig. 5). Here, the amplitude of the biggest initial soliton is 3, and in the Figure maximum of the wave field reaches 3.07. This distinguishes the unipolar gas of the Schamel equation from the unipolar gas of the integrable KdV equation.

The fluctuations of the wave fields, which form after soliton collisions, are small enough in order to influence higher statistical moments:

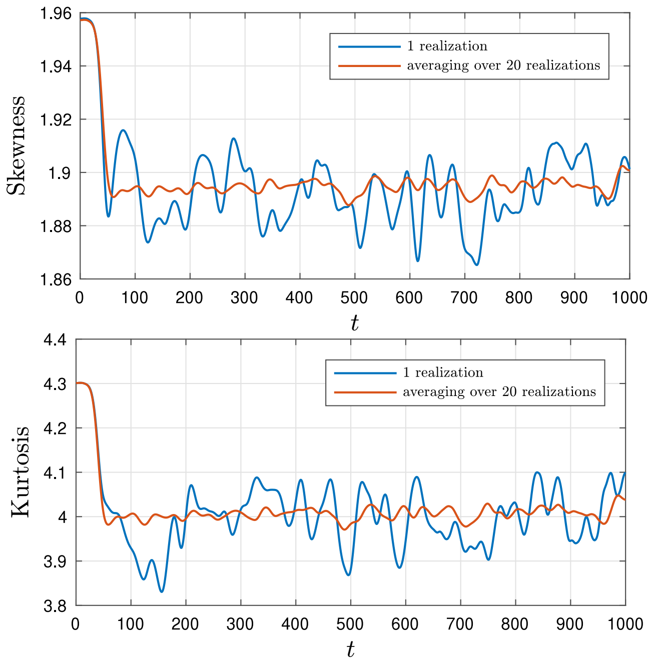

Fig. 6. Temporal evolution of the skewness and kurtosis of the unipolar soliton.

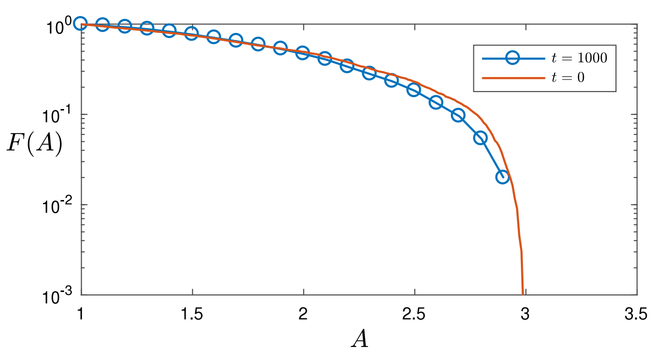

Fig. 7. Distribution function of wave amplitudes at different times averaged over 20 realizations.

skewness and kurtosis. Fig. 6 demonstrates the temporal evolution of skewness and kurtosis of the unipolar soliton field for one realization and averaged value over 20 realizations. Similar to KdV model there is a short transition zone of sharp decrease of moments till about $t = 50$ and it is the same for all realizations of soliton gas. Averaging over realization predictably diminishes the fluctuations of moments and it tends to stationary state.

The interactions among solitons have a discernible impact on the distribution functions of the wave field, as depicted in Fig. 7. Specifically, the average distribution function of wave amplitudes (corresponding to local maxima of the wave field) undergoes a downward shift in the high-amplitude region. Consequently, the presence of large waves diminishes, leading to a more uniform wave field. These findings align with similar observations made in studies involving soliton gases governed by the KdV and mKdV equations [7,25]. The evolution of the number crests count for a single realization at different times is illustrated in Fig. 8.

4. Bipolar soliton

Presence of solitons with different polarity makes the dynamics of the wave system more extreme, because the interaction bipolar solitons increases the maximum of the wave field unlike interaction of unipolar solitons. Formation of abnormally large waves as a result of bipolar soliton collision in the mKdV equation was demonstrated in [7,10,13]. However, because of the presence of radiation the maximum of resulting impulse is less than superposition of amplitudes of interacted solitons in non-integrable systems. The process of bipolar soliton collision within the Schamel equation is presented in Fig. 9.

To study the complex soliton dynamics with different polarities, we set initial wave field as a superposition of separated elevation and depression solitons. Thus we set these solitons to have random amplitudes uniformly distributed in $[1, 3]$ and $[-1, -3]$ respectively. The example of the initial soliton gas is presented in Fig. 10.

Solitons interact with each other over time and there are collisions of big number of solitons. It may contribute to significant growth of the wave field. Thus, in considered realizations the maximum wave field reached almost 7.0 (Fig. 11). The wave field, which contains this freak wave is presented in Fig. 12 and more details of its formation is depicted in Fig. 13.

In the process of nonlinear interaction skewness changes from -0.4 to 0.35 in one realization, but the ensemble of bipolar solitons is close to be symmetrical, and the average value of this statistical moment takes values around zero. Skewness takes both positive and negative values depending on prevalence of positive and negative solitons. Kurtosis being the normalized integral of the wave field in fourth power takes values from 4.3 to 6 in one realization, however the deviation of the averaged value is much less. Moreover, Fig. 14 demonstrates the gradual growth of the averaged value of kurtosis. It may be explained by increasing of number of small amplitude waves (Fig. 15), thus large waves become a "more extreme" and kurtosis gradually increases. This behavior of kurtosis differs from the KdV model, where averaged kurtosis reached the steady state. The increase in kurtosis indicates the amplification of the distribution function tails (Fig. 16).

5. Conclusion

In this study, we have investigated the dynamics of soliton gases using the Schamel equation as our primary framework. Our research involved a numerical exploration of various statistical characteristics, including distribution functions and moments, for both unipolar and bipolar soliton gases. Despite the non-integrability of the Schamel equation, the degree of dispersion generated during soliton interactions remained relatively modest, especially in the unipolar scenario. Consequently, our findings closely aligned with the predictions of the KdV and mKdV equations in the case of unipolar solitons. Nevertheless, in the scenario with bipolar characteristics, we noticed a significant deviation from the mKdV model, especially in terms of kurtosis behavior. The observed rise in kurtosis indicates the enhancement of distribution function tails, indicating the existence of high-amplitude waves.

These novels challenge our previous understanding and highlight the unique characteristics of solitary wave collisions in this particular equation. By unraveling the distinct behavior and properties of solitary wave collisions in the Schamel equation, our study contributes to the broader understanding of solitary wave dynamics. These findings open up new avenues for exploring the intricate dynamics and interactions of solitary waves in non-integrable systems.

CRediT authorship contribution statement

Marcelo V. Flamarion: Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Efim Pelinovsky:** Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Ekaterina Didenkulova:** Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing.

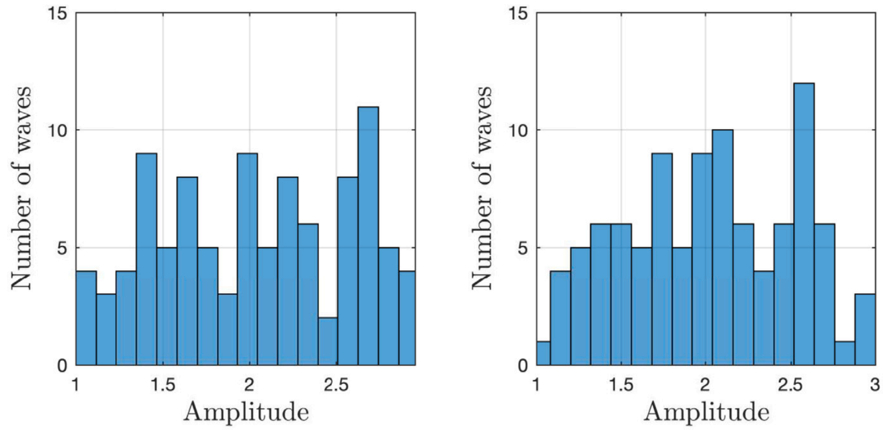

Fig. 8. Histogram of wave amplitude at $t = 0$ (left) and at $t = 1000$ (right).

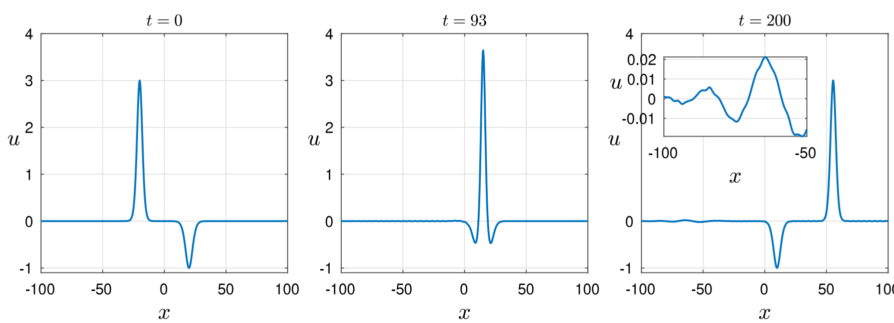

Fig. 9. Interaction process of bipolar soliton interaction within the Schamel equation.

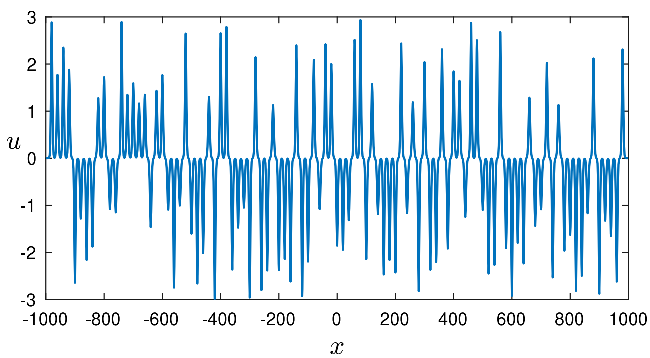

Fig. 10. Initial distribution of bipolar solitons.

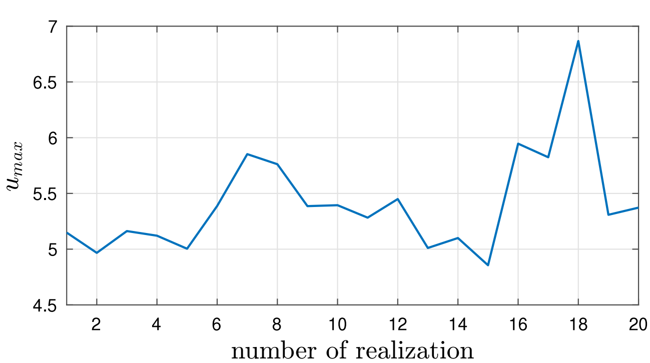

Fig. 11. Maximum of wave amplitudes over time of the bipolar soliton.

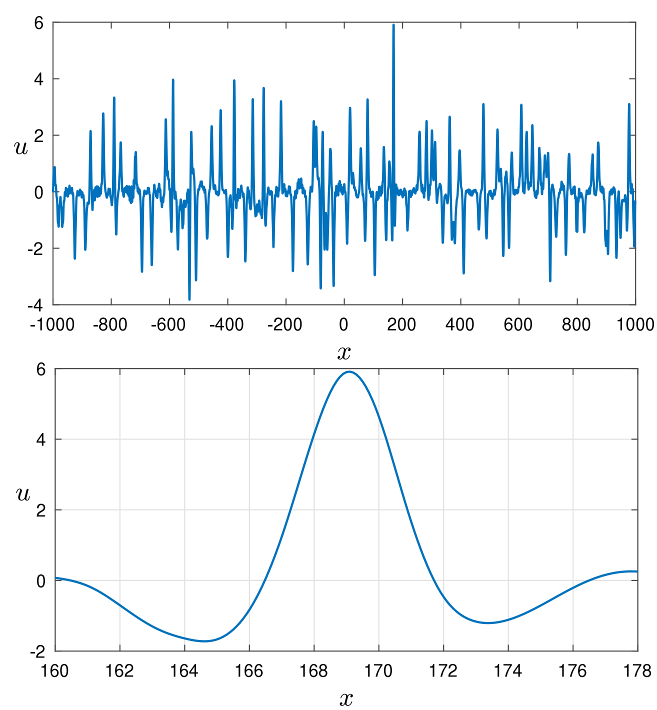

Fig. 12. Wave field of bipolar solitons at $t = 416$. Zoom of the freak wave is on the bottom.

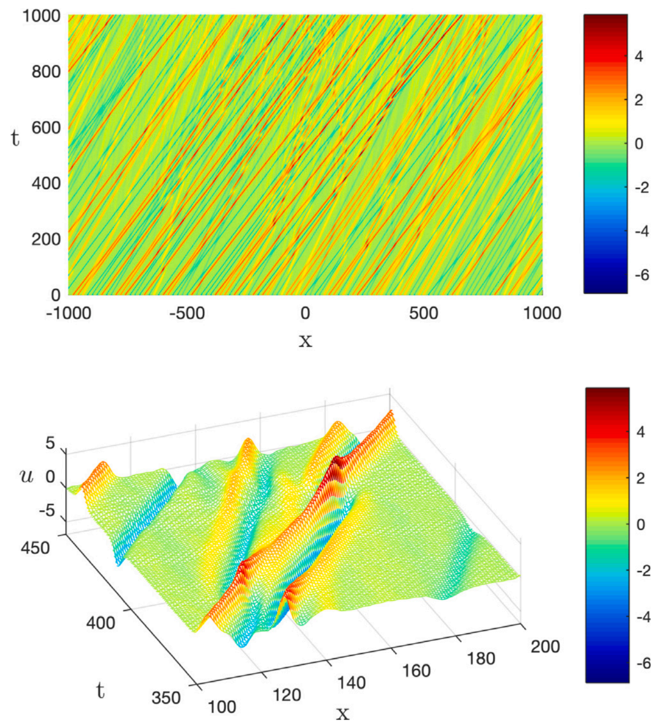

Fig. 13. Spatial–temporal diagram of bipolar soliton gas and zoom in the region of freak wave formation.

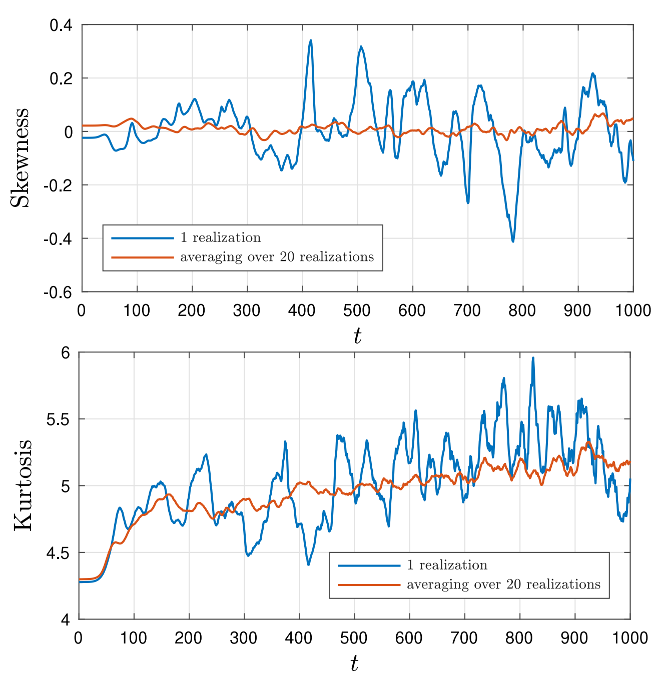

Fig. 14. Temporal evolution of the skewness and kurtosis of the bipolar soliton.

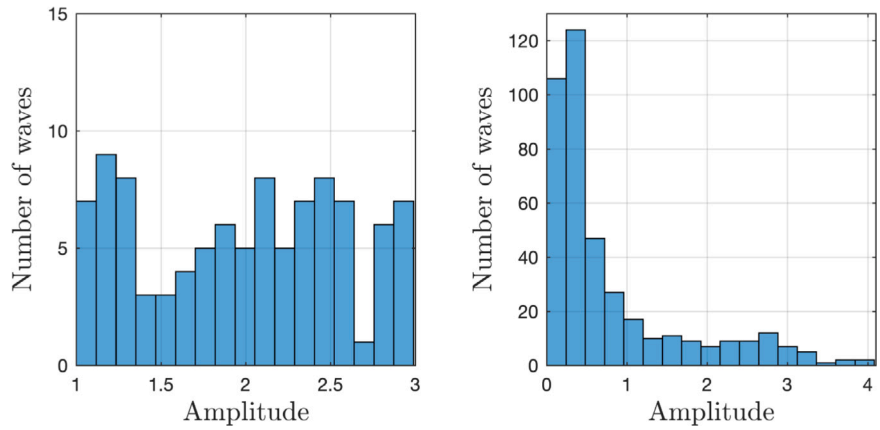

Fig. 15. Histogram of wave amplitude at $t = 0$ (left) and at $t = 1000$ (right).

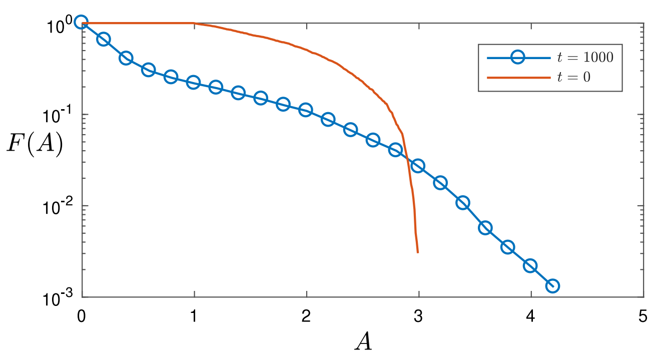

Fig. 16. Distribution function of wave amplitudes at different times averaged over 20 realizations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data sharing is not applicable to this article as all parameters used in the numerical experiments are informed in this paper.

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