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Clustering of passive tracers in a random acoustic velocity field

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ABSTRACT

We consider the effects of passive tracer clustering (e.g., the magnetic field energy in stellar atmospheres) in a random acoustic velocity field. A method for numerical modeling of a two-dimensional random acoustic field is proposed. The field is described by a correlation tensor defined by traveling isotropic waves, taking into account dissipation. Two metrics for measuring the clustering effects are used: concentration and density. Using numerical modeling, we show that the tracer concentration is almost always clustered. The situation with the density is different; as the dissipation tends to zero, the time to reach the clustered states increases significantly. In addition, due to the tracer transport out of the density clustering regions, only a part of the tracer is clustered. For the presented analyses, we considered ensembles of Lagrangian particles and introduced and applied the *statistical topography* methodology.

I. INTRODUCTION

The particle transport in rapidly varying random velocity fields, in particular in wave velocity fields, is an important problem in physics. The effects associated with such motion have important applications in mechanics, hydrodynamics, plasma physics, etc. The study of the problem of tracer clustering in compressible flows began quite a long time ago.[1–5] A review of clustering problems can be found in Ref. 6. More examples of such effects during transport in rapidly varying wave fields include, for example, Fermi acceleration, plasma acceleration, etc. can be found in the literature.[7–9] Moreover, tracer clustering is common in stochastic oceanic and atmospheric flows, i.e., spatial aggregation of various tracers and objects,[10–12] marine ecosystems,[10] formation of clouds,[13] porous media,[14] paleontology,[15] and cosmology.[16]

We focus on the clustering process affected by stochastic acoustic velocity fields. We assume the velocity field to be purely potential (compressible). However, in some cases the clustering of the tracer can occur in velocity fields with a non-vanishing incompressible term as well.

Clustering can be analyzed using the Fokker–Planck equations[17] and with the delta-correlated process approximation, the diffusion approximation, and perturbation theory.[7,9,17–22] These approaches, however, have certain disadvantages. These approximations are quite limiting in terms of applicability and constructing a comprehensive exact theory is unrealistic. Thus, direct numerical modeling makes it possible to verify the results of approximate analytical methods and, even further, to clarify and extend the interpretation of analytical estimates. This is exactly what we do in this work. The goal of this paper is to establish the statistical properties of clustering in random kinematic acoustic flows with dissipation. We use the same approach for numerical modeling as in Refs. 23–25. However, we have modified the method of random velocity field generation to get a correlation tensor for the acoustics wave field and to take into account dissipation.[7,22]

The paper is organized as follows: In Sec. II, we formulate the general problem, equations, characteristics of the random acoustic velocity field, and some formulas of statistical topography. The numerical results and some analysis are presented in Sec. III. The main results are in Sec. IV, followed by discussions and conclusions.

II. PROBLEM FORMULATION

The tracer evolution in random velocity fields can be formulated for both passive and compressible tracers, and the corresponding descriptions fundamentally differ, thus reflecting profoundly different clustering properties. We introduce the passive-tracer concentration $C(\mathbf{r}, t)$ and floating-tracer density $\rho(\mathbf{r}, t)$, which are the main fields of interest, both varying in space and evolving in time. Second, let us discuss the differences between C and ρ and explain the terminology. The passive tracer is just a fluid particle and its dynamics is subject to the standard continuity equation for material tracers; in turn, this equation can be restricted to describe the evolution of C. The concentration C is conserved for each material particle. However, a compressible tracer is not passive in the sense that it is not a fluid particle, and its density changes as the fluid particle moves in the flow due to the velocity divergence. The latter effect can be viewed as the compressibility of the floating-tracer density ρ, hence "density" for a compressible tracer vs "concentration" for a passive (incompressible) tracer. As we will see below, particle motion in a random acoustic velocity field affects concentration and density clustering differently. Thus, we will study the particle dynamics and their density by solving numerically the continuity equations for the concentration of particles and their density. We do not take into account the interaction between particles.

To describe the motion of particles in a random velocity field, the following first-order differential equation is usually used:[7,22]

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{u}(\mathbf{r}, t), \quad \mathbf{r}(0) = \mathbf{r}_0. \tag{1}$$

Below we will consider the two-dimensional case and set $\mathbf{u}(\mathbf{r}, t) = (u(\mathbf{r}, t), v(\mathbf{r}, t))$ and $\mathbf{r} = (x, y)$.

The continuity equations governing the tracer concentration C and the tracer density field ρ in the random velocity field $\mathbf{u}(\mathbf{r}, t)$ are as follows:[25–28]

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{r}, t)\frac{\partial}{\partial \mathbf{r}}\right)C(\mathbf{r}, t) = 0\,, \quad C(\mathbf{r}, 0) = C_0(\mathbf{r})\,; \tag{2}$$

$$\frac{\partial}{\partial t}\rho(\mathbf{r}, t) + \frac{\partial}{\partial \mathbf{r}}\mathbf{u}(\mathbf{r}, t)\rho(\mathbf{r}, t) = 0, \quad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r}). \tag{3}$$

We do not consider the influence of dynamic diffusion. Here, $C_0(\mathbf{r})$ is the initial distribution of the tracer concentration and $\rho_0(\mathbf{r})$ is the initial distribution of the tracer density, and the stochastic properties of Eqs. (1)–(3) are determined by the random two-dimensional velocity field $\mathbf{u}(\mathbf{r}, t)$.

Evolution of each Lagrangian particle is governed by 1 and its density and concentration are governed by the following equations:

$$\begin{aligned} &\frac{d\rho(t;\boldsymbol{\xi})}{dt} = -\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial \mathbf{r}}\rho(t;\boldsymbol{\xi}), \quad \rho(0;\boldsymbol{\xi}) = \rho_0(\boldsymbol{\xi}); \\ &\frac{dC(t;\boldsymbol{\xi})}{dt} = 0, \quad C(0;\boldsymbol{\xi}) = C_0(\boldsymbol{\xi}), \end{aligned} \tag{4}$$

where $\boldsymbol{\xi}$ is the initial position of the trajectory. The Eulerian density and concentration fields are defined along the following trajectories:

$$\rho(\mathbf{r}, t) = \rho(t;\boldsymbol{\xi}(\mathbf{r};t)); \tag{5}$$

$$C(\mathbf{r}, t) = C(t;\boldsymbol{\xi}(\mathbf{r};t)). \tag{6}$$

A. Statistical properties of the random velocity field

We consider the motion of tracer particles inside a random Gaussian acoustic velocity field $\mathbf{u}(\mathbf{r}, t)$ ($\mathrm{div}\mathbf{u}(\mathbf{r}, t) \neq 0$), statistically homogeneous and isotropic in space, as well as stationary in time, with correlation and spectral tensors ($\tau = t - t'$, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$)

$$\langle u_i(\mathbf{r}, t)u_j(\mathbf{r}', t')\rangle = \sigma_{\mathbf{u}}^2 B_{ij}(\mathbf{R}, \tau) = \sigma_{\mathbf{u}}^2 \int dk_x dk_y E_{ij}(k_x, k_y) f(\mathbf{k}\cdot\mathbf{R}, \tau). \tag{7}$$

Here, $\sigma_{\mathbf{u}}^2 = \langle \mathbf{u}(\mathbf{r}, t)^2\rangle$ is the dispersion of the velocity field, and the function[7]

$$f(\mathbf{k}\cdot\mathbf{R}, \tau) = e^{-\lambda(k)\tau}\cos(\mathbf{k}\cdot\mathbf{R} - \omega(k)\tau) \tag{8}$$

is responsible for the wave properties of the velocity field, and the indices i and j correspond to the coordinates in space. We will use the dispersion relation for acoustic waves $\omega = \omega(k) = kc$, where $\mathbf{k} = (k_x, k_y)$, $k = \sqrt{k_x^2 + k_y^2} = |\mathbf{k}|$, and c is the speed of sound. Exponentially decaying terms are responsible for the dissipation of acoustic waves. We will consider the attenuation coefficient in the form

$$\lambda(k) = \lambda_p k^2. \tag{9}$$

The spectral density $E_{ij}(k_x, k_y)$ we choose is in the form

$$E_{ij}(k_x, k_y) = E_i(k_x, k_y)E_j(k_x, k_y) = E(\mathrm{k})\frac{k_i k_j}{k^2}, \tag{10}$$

where $E_i(k_x, k_y) = \sqrt{E(\mathrm{k})}\frac{k_i}{k}$ and $E(k)$ will be defined below. This spectral density corresponds to the potential velocity field.

To numerically implement such a velocity field, we consider the following equation:[7]

$$\begin{aligned} &\frac{d}{dt}v(\mathbf{k}, \mathbf{r}, t) + \lambda(k)v(\mathbf{k}, \mathbf{r}, t) = E_j(k_x, k_y)c_j(\mathbf{k}, t)e^{-i\mathbf{k}\cdot\mathbf{R}}; \\ &v(\mathbf{k}, \mathbf{r}, 0) = v_o(\mathbf{k}, \mathbf{r}), \end{aligned} \tag{11}$$

which we modify as follows:

$$\begin{aligned} &\frac{d}{dt}v_j(\mathbf{k}; t) + (\lambda(k) \pm i\omega(k))v_j(\mathbf{k}; t) = bE_j(k_x, k_y)c_j(\mathbf{k}, t); \\ &v_j(\mathbf{k}; 0) = v_{oj}(\mathbf{k})E_j(k_x, k_y). \end{aligned} \tag{12}$$

Thus, if we multiply the solution of this equation by $e^{-i\mathbf{k}\cdot\mathbf{R}}$ we will consider a wave with dissipation propagating in the direction defined by $\mathbf{k}$ (or opposite direction due to the sign before $\omega(k)$. We will assume that the sources $c_j(\mathbf{k}, t)$ are delta-correlated stochastic functions in time and the wave vector with statistically independent components and initial condition $v_{oj}(\mathbf{k})$ is delta-correlated with statistically independent components.[23,24] We will obtain a random wave field as a solution of Eq. (12) and try to select a constant b so that it has the desired correlation tensor.

The formal solution of Eq. (12) has the form [for the case $(\lambda(k) - i\omega(k))$]

$$\begin{aligned} v_j(\mathbf{k}; t) = {}& v_{oj}(\mathbf{k})E_j(k_x, k_y)e^{-(\lambda(k) - i\omega(k))t} \\ &+ b\int_0^t d\theta E_j(k_x, k_y)c_j(\mathbf{k}, \theta)e^{-(\lambda(k) - i\omega(k))(t-\theta)}. \end{aligned} \tag{13}$$

Let us choose constant and initial conditions to obtain the desired correlation matrix. After multiplying by $e^{-i\mathbf{k}\cdot\mathbf{r}}$ and $e^{-i\mathbf{k}'\cdot\mathbf{r}'}$, one gets

$$\left\langle v_i(\mathbf{k};t)e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\bar{v}_j(\mathbf{k}';t')\right\rangle = E_i\left(k_x,k_y\right)E_j\left(k'_x,k'_y\right)e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\left\langle\left(v_{oi}(\mathbf{k})e^{-\left(\lambda(k)-i\omega(k)\right)t}+b\int_0^t d\theta c_i(\mathbf{k},\theta)e^{-\left(\lambda(k)-i\omega(k)\right)(t-\theta)}\right)\right.$$
$$\left.\times\left(\bar{v}_{oj}(\mathbf{k}')e^{-\left(\lambda(k')+i\omega(k')\right)t'}+\bar{b}\int_0^{t'} d\theta'\bar{c}_j(\mathbf{k}',\theta')e^{-\left(\lambda(k')+i\omega(k')\right)(t'-\theta')}\right)\right\rangle$$
$$= E_i\left(k_x,k_y\right)E_j e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'}\left(k'_x,k'_y\right)\left(\left\langle v_o(\mathbf{k})\bar{v}_o(\mathbf{k}')e^{-\lambda(k)t-\left(\lambda(k')\right)t'}e^{i\omega(k)(t-t')}\right\rangle\right.$$
$$\left.+\left\langle b\bar{b}\int_0^{t,t'} d\theta d\theta' c_i(\mathbf{k},\theta)\bar{c}_j(\mathbf{k}',\theta')e^{-(\lambda(k))(t-\theta)-\left(\lambda(k')\right)(t'-\theta')}e^{i\omega(k)(t-\theta)-i\omega(k')(t'-\theta\prime)}\right\rangle\right). \tag{14}$$

The notations $\bar{v}$, $\bar{b}$, and $\bar{c}$ mean the complex conjugates here and below. We have omitted the cross terms since they will vanish during averaging since we assume the statistical independence of the initial condition from the integrand. Furthermore, we consider b as the deterministic constant and also assume the delta correlation of $v_o(\mathbf{k})$ of wave vector components and $c_i(\mathbf{k},\tau)$ in wave vector and time. We obtain

$$\left\langle v_i(\mathbf{k};t)e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\bar{v}_j(\mathbf{k}';t')\right\rangle = E_i\left(k_x,k_y\right)E_j\left(k'_x,k'_y\right)e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\left\langle\left(v_{oi}(\mathbf{k})e^{-\left(\lambda(k)-i\omega(k)\right)t}+b\int_0^t d\theta c_i(\mathbf{k},\theta)e^{-\left(\lambda(k)-i\omega(k)\right)(t-\theta)}\right)\right.$$
$$\left.\times\left(\bar{v}_{oj}(\mathbf{k}')e^{-\left(\lambda(k')+i\omega(k')\right)t'}+\bar{b}\int_0^{t'} d\theta'\bar{c}_j(\mathbf{k}',\theta')e^{-\left(\lambda(k')+i\omega(k')\right)(t'-\theta')}\right)\right\rangle$$
$$= E_i\left(k_x,k_y\right)E_j\left(k'_x,k'_y\right)e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'}\left(\left\langle v_o(\mathbf{k})\bar{v}_o(\mathbf{k}')e^{-\lambda(k)t-\left(\lambda(k')\right)t'}e^{i\omega(k)(t-t')}\right\rangle\right.$$
$$\left.+\left\langle b\bar{b}\int_0^{t,t'} d\theta d\theta' c_i(\mathbf{k},\theta)\bar{c}_j(\mathbf{k}',\theta')e^{-(\lambda(k))(t-\theta)-\left(\lambda(k')\right)(t'-\theta')}e^{i\omega(k)(t-\theta)-i\omega(k')(t'-\theta')}\right\rangle\right). \tag{15}$$

Now, using the properties of the delta function and assuming that $t'\le t$, we reduce the integral to one dimension as follows:

$$\begin{aligned}
&\left\langle v_i(\mathbf{k};t)e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\bar{v}_j(\mathbf{k}';t')\right\rangle\\
&= \sigma_v^2\delta(\mathbf{k}-\mathbf{k}')E_{ij}\left(k_x,k_y\right)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{i\omega(k)(t-t')}\\
&\quad\times\left(e^{-\lambda(k)(t+t')}+b\bar{b}\int_0^{t'}d\theta e^{-(\lambda(k))(t+t'-2\theta)}\right)\\
&= \sigma_v^2\delta(\mathbf{k}-\mathbf{k}')E_{ij}\left(k_x,k_y\right)e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{i\omega(k)(t-t')}\\
&\quad\times\left(e^{-\lambda(k)(t+t')}+|b|^2\frac{e^{-(\lambda(k))(t-t')}-e^{-(\lambda(k))(t+t')}}{2\lambda(k)}\right).
\end{aligned}\tag{16}$$

Let us choose a real constant $b=\sqrt{2\lambda(k)}$ and get

$$\begin{aligned}
\langle u_i(\mathbf{r},t)u_j(\mathbf{r}',t')\rangle &= \sigma_{\mathbf{u}}^2 B_{ij}(\mathbf{R},\tau)\\
&= \sigma_{\mathbf{u}}^2\int dk_x dk_y\left|E_{ij}\left(k_x,k_y\right)\right|^2 e^{-\lambda(k)\tau}e^{-i\mathbf{k}\cdot\mathbf{R}+i\omega(k)\tau}.
\end{aligned}\tag{17}$$

Now if we take the statistically independent solutions of type (13) for the case $(\lambda(k)+i\omega(k))$, we obtain a correlation tensor of the form (7).

Thus, for the numerical implementation, we will consider the spectral representation of the velocity field, including the integral representation over time (an analogue of Duhamel's formula), as

$$\begin{aligned}
u_j(\mathbf{r},t)=&\frac{\sigma_{\mathbf{u}}}{\sqrt{2}}\int dk_x dk_y E_j\left(k_x,k_y\right)e^{-i\mathbf{k}\cdot\mathbf{r}}\left[c_j^1(\mathbf{k})e^{-\left(\lambda(k)-i\omega(k)\right)t}\right.\\
&+c_j^3(\mathbf{k})e^{-\left(\lambda(k)+i\omega(k)\right)t}+\sqrt{2\lambda(k)}\left(\int_0^t d\tau c_j^2(\mathbf{k},\tau)e^{-\left(\lambda(k)-i\omega(k)\right)(t-\tau)}\right.\\
&\left.\left.+\int_0^t d\tau c_j^4(\mathbf{k},\tau)e^{-\left(\lambda(k)+i\omega(k)\right)(t-\tau)}\right)\right],
\end{aligned}\tag{18}$$

here $j=x,y$. Here, we have added the indices α to indicate that $c_j^\alpha(\mathbf{k},t)$ and $c_j^\alpha(\mathbf{k})$ are statistically independent random functions that are delta-correlated in all their arguments, i.e.,

$$\left\langle c_j^\alpha(\mathbf{k}',t')c_l^\beta(\mathbf{k}'',t'')\right\rangle = \delta_{\alpha\beta}\delta(\mathbf{k}'-\mathbf{k}'')\delta(t'-t''), \quad (\alpha,\beta=2,4);$$
$$\left\langle c_j^\alpha(\mathbf{k}')c_l^\beta(\mathbf{k}'')\right\rangle = \delta_{\alpha\beta}\delta(\mathbf{k}'-\mathbf{k}''), \quad (\alpha,\beta=1,3). \tag{19}$$

Note that random sequences (generally speaking functions) are chosen to be identical for the components x and y, therefore, there is no δ_{jl} in formula (19).

So, by generating numerically random sequences, $c_j^\alpha(\mathbf{k},t)$, satisfying relations (19) and calculating the integrals in (18), we obtain a numerical implementation of a random velocity field having a correlation tensor of the form (7). Such a velocity field corresponds to a random acoustic field with dissipation and has a finite correlation radius in time.

Let us choose the spectral density $E_{ij}(k_x,k_y)$ corresponding to the potential velocity field

$$E_{ij}(k_x,k_y) = E(\mathrm{k})\frac{k_ik_j}{k^2}, \quad E(\mathrm{k}) = \frac{l^2}{(4\pi)^2}\exp\left\{-\frac{l^2k^2}{2}\right\}. \tag{20}$$

B. Statistical topography of random fields

The convenient way to analyze the clustering effects of buoyant tracers in compressible flows is statistical topography.[17,23,24,29] Let us introduce the following Liouville indicator function:

$$\varphi(\mathbf{R},t;\rho^*) = \delta(\rho(\mathbf{R},t)-\rho^*). \tag{21}$$

Then, we consider the function

$$S(t;\rho^*) = \int d\mathbf{R}\theta(\rho(\mathbf{R},t)-\rho^*) = \int d\mathbf{R}\int_{\rho^*}^{\infty} d\rho'\varphi(\mathbf{R},t;\rho'), \tag{22}$$

which is the area of regions in which the random tracer density exceeds a given level ρ^*, and similarly, the function

$$M(t;\rho^*) = \int d\mathbf{R}\rho(\mathbf{R},t)\theta(\rho(\mathbf{R},t)-\rho^*) = \int d\mathbf{R}\int_{\rho^*}^{\infty} d\rho'\rho'\varphi(\mathbf{R},t;\rho'), \tag{23}$$

which is the mass of tracer concentrated in these areas. Here, $\theta(\rho(\mathbf{R},t)-\rho^*)$ is the Heaviside theta function.

Next, we take advantage of the fact that the average of the indicator function is $P(\mathbf{R},t;\rho^*) = \langle\delta(\rho(\mathbf{R},t)-\rho^*)\rangle$, a one-time probability density in time and one-point in space.[17,24,29] By averaging Eqs. (22) and (23) over an ensemble of velocity field realizations, we obtain the following expressions:

$$\langle S(t;\rho^*)\rangle = \int d\mathbf{R}\int_{\rho^*}^{\infty} d\rho' P(\mathbf{R},t;\rho'),$$
$$\langle M(t;\rho^*)\rangle = \int d\mathbf{R}\int_{\rho^*}^{\infty} d\rho'\rho' P(\mathbf{R},t;\rho'). \tag{24}$$

In the case of a spatially homogeneous density field $\rho(\mathbf{R},t)$, the one-point probability density does not depend on $\mathbf{R}$ and expressions in (24) are simplified as follows:

$$\langle s_{\mathrm{hom}}(t;\rho^*)\rangle = \langle\theta(\rho(\mathbf{R},t)-\rho^*)\rangle = \mathrm{P}\{\rho(\mathbf{R},t)>\rho^*\} = \int_{\rho^*}^{\infty}\mathrm{d}\rho'\mathrm{P}(t;\rho'),$$
$$\langle m_{\mathrm{hom}}(t;\rho^*)\rangle = \int_{\rho^*}^{\infty} d\rho'\rho' P(t;\rho'). \tag{25}$$

Here, $s_{\mathrm{hom}}(t;\rho^*)$ and $m_{\mathrm{hom}}(t;\rho^*)$ are the specific values, i.e., referred to as unit area.[17,24]

For a positive random density field, clustering conditions with probability 1 (i.e., in almost any realization) lead to the following asymptotic expressions:

$$\langle s_{\mathrm{hom}}(t;\rho^*)\rangle \to 0, \quad \langle m_{\mathrm{hom}}(t;\rho^*)\rangle \to 1. \tag{26}$$

These expressions mean that the area of regions with a density exceeding a given level tends to 0, and the tracer mass concentrated in these regions (clusters) tends to 1.

At larger times as compared to the diffusion times, one can use the estimates of[17,24,30]

$$\langle s_{\mathrm{hom}}(t,\rho^*)\rangle = \mathrm{P}\{\rho(\mathbf{R},t)>\rho^*\} \approx \sqrt{\frac{\rho_0}{\pi\rho^* t/\tau}}\mathrm{e}^{-\frac{1}{4}\frac{t}{\tau}},$$
$$\langle m_{\mathrm{hom}}(t,\rho^*)\rangle/\rho_0 \approx 1-\sqrt{\frac{\rho^*}{\pi\rho_0 t/\tau}}e^{-\frac{1}{4}\frac{t}{\tau}}. \tag{27}$$

Here, $\tau = 1/D$ and D is the corresponding diffusion coefficient.

Even though these estimates are valid only for a potential random velocity field without a regular component, they can be a good reference basis for estimating the velocity and degree of clustering of the tracer. We will compare our numerical results with these expressions. In Refs. 7 and 22, the evolution of the tracer and a magnetic field in random wave fields were studied. In particular, for an acoustic random velocity field, it was shown that the Stokes drift of particles and density clustering take place. These processes occur on different time scales, and it has been suggested that they can be separated. However, we will show below that this is not always possible.

In Ref. 22, an estimate of the typical realization[7,31] of the tracer density field in a random acoustic velocity field was obtained in the form of the exponential function with the Lyapunov exponent expressed as

$$-\alpha t = -\frac{1}{2}Dt = -\frac{1}{2}\left(\frac{\sigma_{\mathbf{u}}^2}{c^2}\int dk_x dk_y k^2\frac{\lambda_p}{\lambda_p^2k^2+c^2}E(k)\right)t. \tag{28}$$

Thus, this work shows that the curve of a typical realization decays exponentially, when there is non-zero wave dissipation. Thus, in each realization of the stochastic velocity field, the tracer must be clustered with probability one. The clustering rate is determined by the diffusion coefficient D, which has two asymptotics as follows:

$$D \sim \frac{\lambda_p}{c^2} \quad for \quad \lambda_p \ll cl;$$
$$D \sim \frac{1}{\lambda_p} \quad for \quad \lambda_p \gg cl. \tag{29}$$

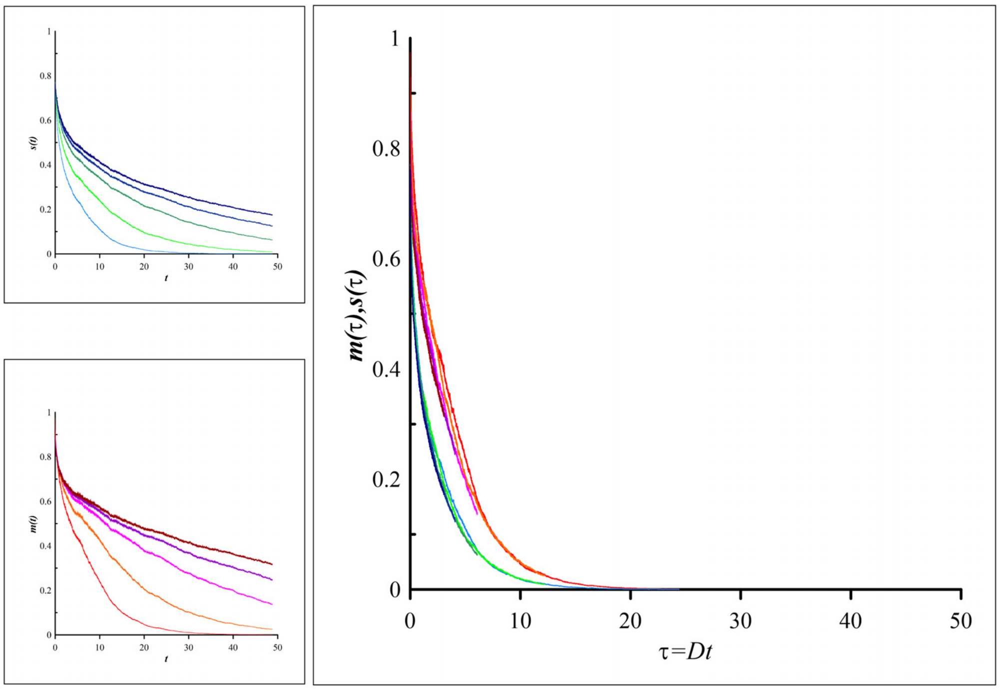

FIG. 1. Dependence of the mass accumulated in clusters (lower left frame) and clustered area (upper left frame) on the calculation time, the same dependencies on diffusion time $\tau = Dt$ (right frame). $\sigma_{\mathbf{u}} = 1$, $l = 0.02 = 4h$. The colors of the curves correspond to different values of wave field dissipation: $\frac{\lambda_p}{cl} = 0.8$—dark blue, ruby red; $\frac{\lambda_p}{cl} = 0.5$—navy blue, purple; $\frac{\lambda_p}{cl} = 0.4$—Kentucky green, magenta; $\frac{\lambda_p}{cl} = 0.2$—green, orange; and $\frac{\lambda_p}{cl} = 0.1$—dodger blue, red.

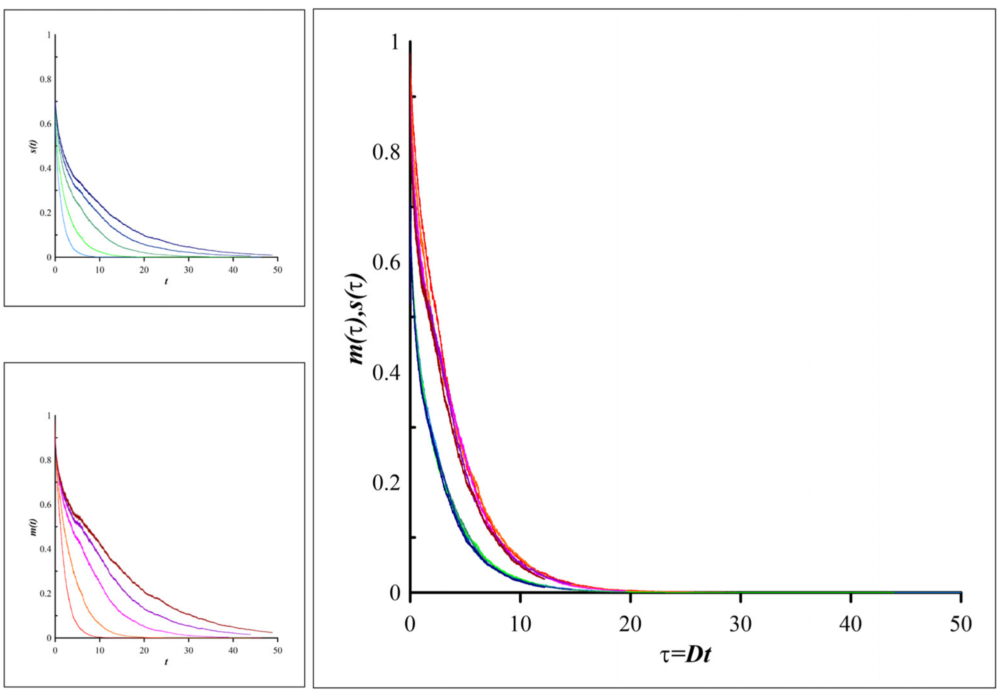

FIG. 2. Dependence of the mass accumulated in clusters (lower left frame) and clustered area (upper left frame) on the calculated time, the same dependencies on diffusion time $\tau = Dt$ (right frame). $\sigma_{\mathbf{u}} = 2$, $l = 0.02 = 4h$. The colors of the curves correspond to the same wave field dissipation values as in Fig. 1.

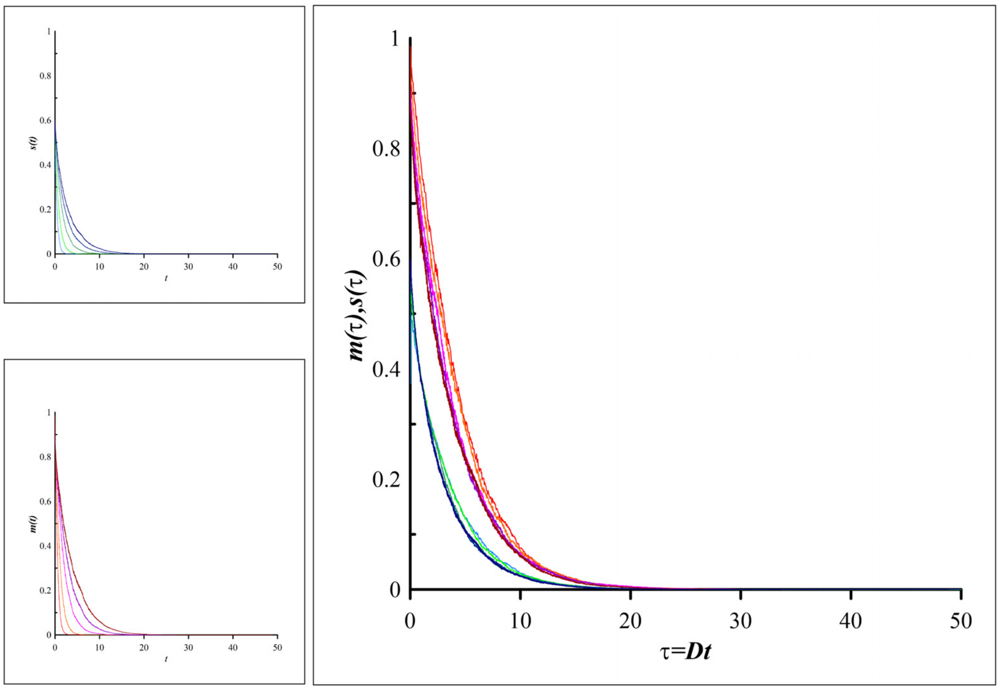

FIG. 3. The same as Figs. 1 and 2 but for $\sigma_{\mathbf{u}} = 4$, $l = 0.02 = 4h$.

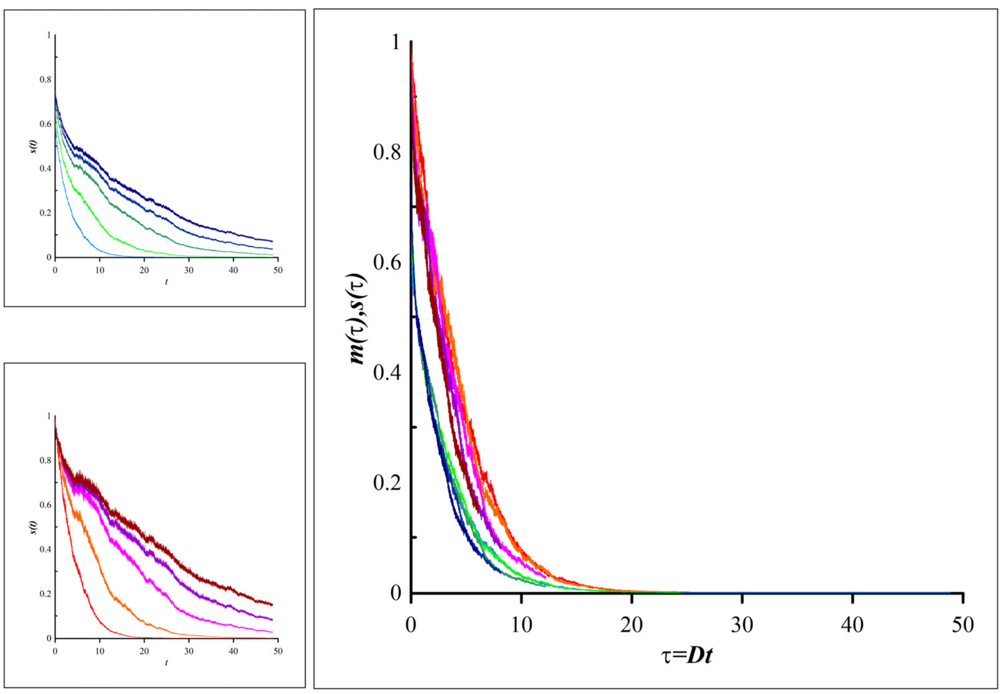

FIG. 4. The same as Figs. 1 and 2 but for $\sigma_{\mathbf{u}} = 2$, $l = 0.04 = 8h$.

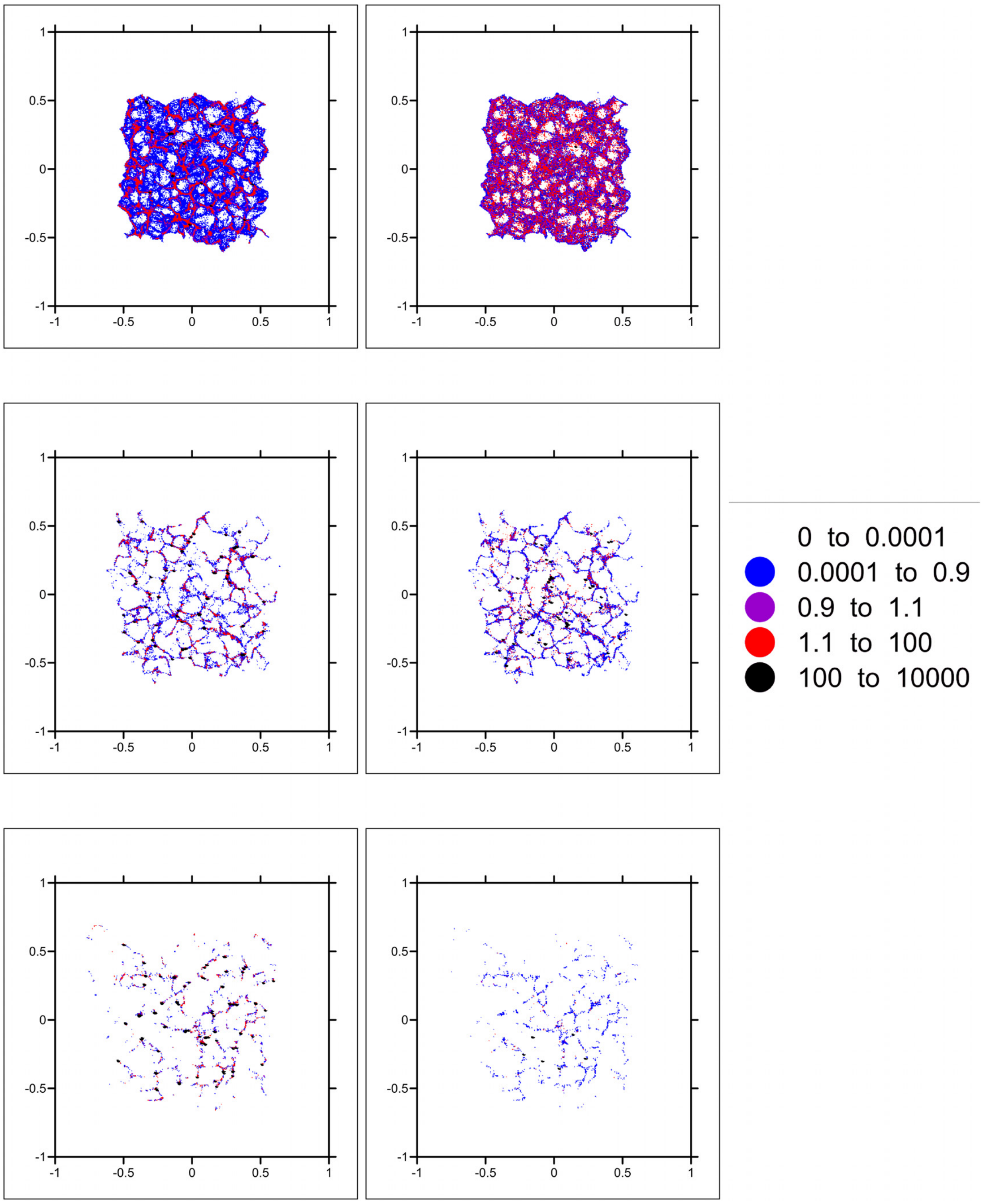

FIG. 5. Distribution of the tracer density (right) and concentration of tracer particles (left). $\sigma_{\mathbf{u}} = 1$, $l = 0.02 = 4h$, $\frac{\lambda_p}{cl} = 0.1$. The distributions are given for the following times (from top to bottom): $Dt = 2.4414, 9.7656, 19.5312$.

We will consider the case $\lambda_p \sim cl$ in a numerical model and try to understand whether clustering of the tracer takes place and how the diffusion coefficient depends on dissipation. To calculate the density dynamics and simulate a discrete analogue of the velocity field, we will use the methods from Ref. 24.

III. NUMERICAL RESULTS

Let us consider the evolution of a tracer area with a constant density $\rho = 1$ in the form of a square of 200×200 grid cells, with a spatial cell size of $h = 0.005$. We use the complete computational domain in which the random acoustic field is generated with dimensions of

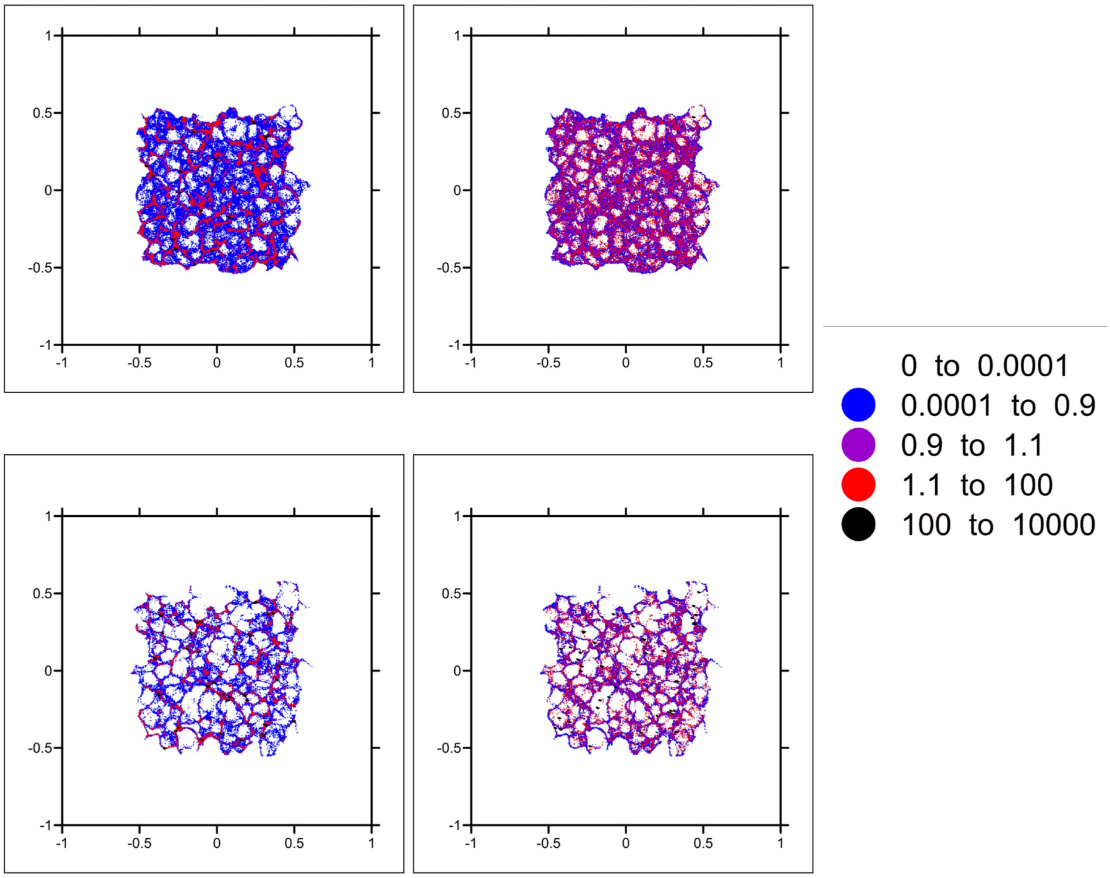

FIG. 6. Distribution of the tracer density (right) and concentration of tracer particles (left). $\sigma_{\mathbf{u}} = 1$, $l = 0.02 = 4h$, $\frac{\lambda_p}{cl} = 0.4$. The distributions are given for the following times (from top to bottom): $Dt = 2.4414, 4.8828$.

2048×2048 grid cells. In each grid cell, 100 particles are evenly distributed, and the total number of particles in the spot is 4 000 000. The time step was chosen equal to the space step $\Delta t = 0.005$. We choose the velocity scale equal to 1 and set $c = 1$. Accordingly, λ_p has a characteristic scale ch. We will choose sufficiently large values of the dissipation coefficient so that the estimate $\frac{\lambda_p}{cl} \leq 1$ is valid. Note that we use the space and time step equal to 1 when we generate a random velocity field and then we scale it to $h = 0.005$ and $\Delta t = 0.005$ to calculate particle trajectories.

Equation (1) and the equation for ρ from (4) are integrated in time using the standard Euler–Ito scheme.[32] Concentration does not change along the trajectory; thus, we use the number of particles per area as the concentration.

First of all, we analyze the characteristics of statistical topography. Figures 1–4 show the dependencies of the area of clusters and the mass accumulated in them. The first conclusion is that the expression for the diffusion coefficient follows from formulas (28) and (29) obtained in[22]

$$D = \frac{\sigma_{\mathbf{u}}^2}{c^2} \int dk_x dk_y k^2 \frac{\lambda_p}{\lambda_p^2 k^2 + c^2} E(k) \approx \tilde{D} \frac{\lambda_p \sigma_{\mathbf{u}}^2}{c^2} \quad \mathit{for} \quad \lambda_p \ll cl, \quad (30)$$

and is confirmed by the numerical simulation. Here, $\tilde{D}$ is constant of order one. Note that expression (30) for the diffusion coefficient remains valid for dissipation $\lambda_p \sim cl$.

The second, and rather unexpected, conclusion is that formally the criterion for the presence of clustering (26) is not met, which at first glance contradicts the results of the previous work.[22] However, an analysis of the spatial distribution of the tracer density and tracer particles allows us to understand the reason for this effect. These distributions, for the same parameter values as in Figs. 1–4, are shown in Figs. 5–8.

In these figures, the left column shows the distribution of the tracer density, and the right column shows the concentration of tracer particles. Comparing the right and left columns shows that areas of high particle concentration and areas of high tracer density do not coincide. Moreover, at sufficiently long times, almost all clusters (regions with a high tracer density occupying a very small area) are located in areas in which there is practically no tracer. Thus, we can conclude that the tracer transport by a random acoustic field and the tracer clustering in this field lead to the following: the tracer is transferred to areas in which the density decreases and, accordingly, there is very little tracer in areas where clustering occurs.

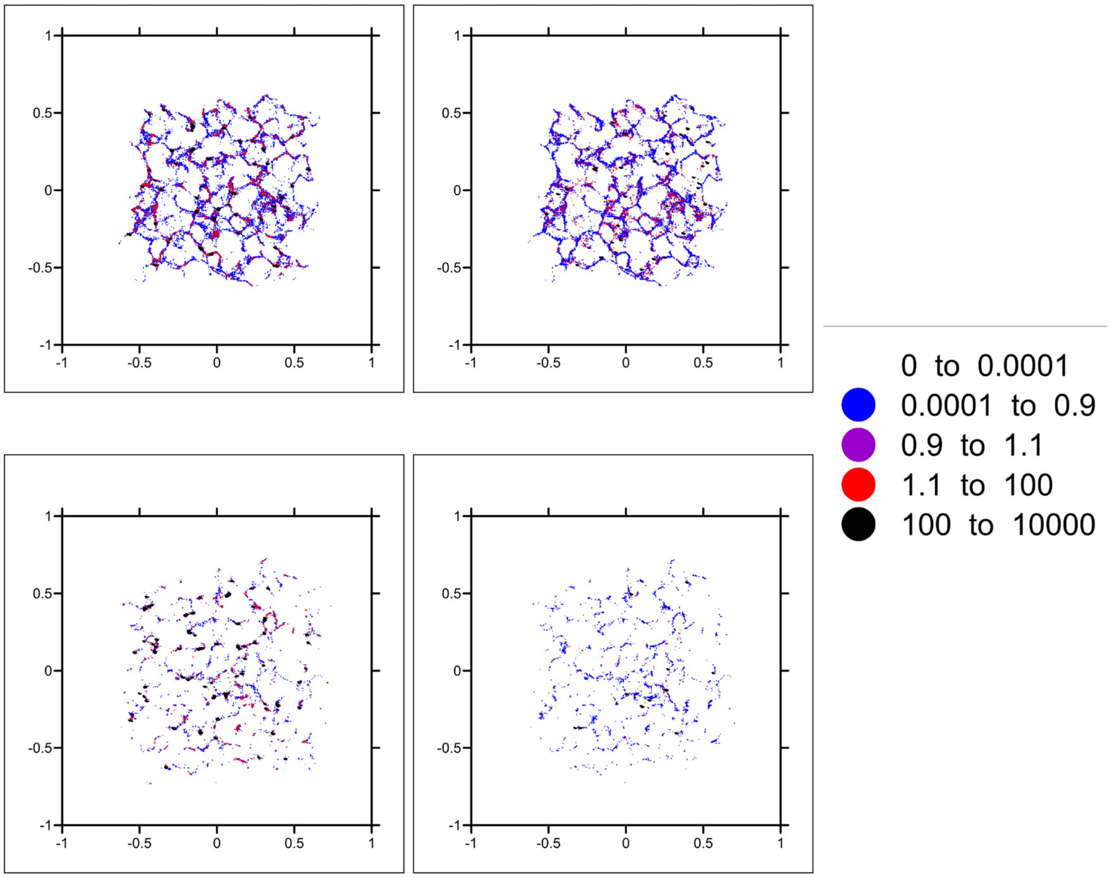

FIG. 7. Distribution of the tracer density (right) and concentration of tracer particles (left). $\sigma_{\mathbf{u}} = 2$, $l = 0.02 = 4h$, $\frac{\lambda_p}{cl} = 0.1$. The distributions are given for the following times (from top to bottom): $Dt = 9.7656, 19.5312$.

In Ref. 33, it has been shown that clusters exist for a finite time, then quickly disintegrate and form again. Thus, we see that clustering takes place, but since the tracer is transported away from the areas of high density, in particular after the destruction of clusters, only a small part of the tracer can be actually clustered. These remaining clusters also have only finite densities.

For clarity, Fig. 9 shows the distributions of tracer density (red and purple) and particle concentration (blue and black) in one figure. We can see here that the tracer is concentrated along certain lines (we can say that this is quasi-clustering because of only finite density), and clusters are formed in the areas where there are very few particles. Note that the small number of particles that remain outside the areas of tracer concentration are almost completely contained in clusters (regions with a high tracer density).

Thus, the result of the work[7] should be clarified. The authors used the Fokker–Plank equation to obtain the probability density of the particle density ρ. This equation has two terms with two different diffusion coefficients. One of the term corresponds to the lognormal probability density and it means that the clustering must take place. The second term corresponds to average particle transport (the Stokes drift), and the probability distribution of the position of Lagrangian particles becomes anisotropic.

In the current study, we confirm that the conditions for tracer clustering in a random acoustic velocity field are realized. The numerically obtained diffusion coefficients have the same dependencies on the random field parameters (dispersion, spatial correlation radius, and dissipation) as those obtained analytically in the previous papers.[7,22] Moreover, the average transport of particles (Stokes drift) occurs, which hinders effective clustering. However, the conclusion that the influence of transport can be ignored is not entirely correct. In fact, the clusters have a fairly long lifetime compared to the time of their formation or decay.[33] Thus, over several cycles of cluster formation and decay, tracer transport has a significant impact.

We believe that our results can also be useful for analyzing traffic in megacities. This could be one direction for further research. Also, we have plans to consider different random wave fields in the future. For example, we can use the dispersion relations for the gravity waves $\omega(\mathbf{k}) = \sqrt{gk}$ or for internal waves $\omega(\mathbf{k}) = N\sqrt{k^2 - k_z^2}/k$. In the last case, N is Brent Weissäl's frequency for Rossby waves in the atmosphere or ocean, and we need to consider a 3D random velocity field.

IV. CONCLUSIONS

We carried out numerical simulations of the transport and clustering of the tracer in a random acoustic velocity field. The results

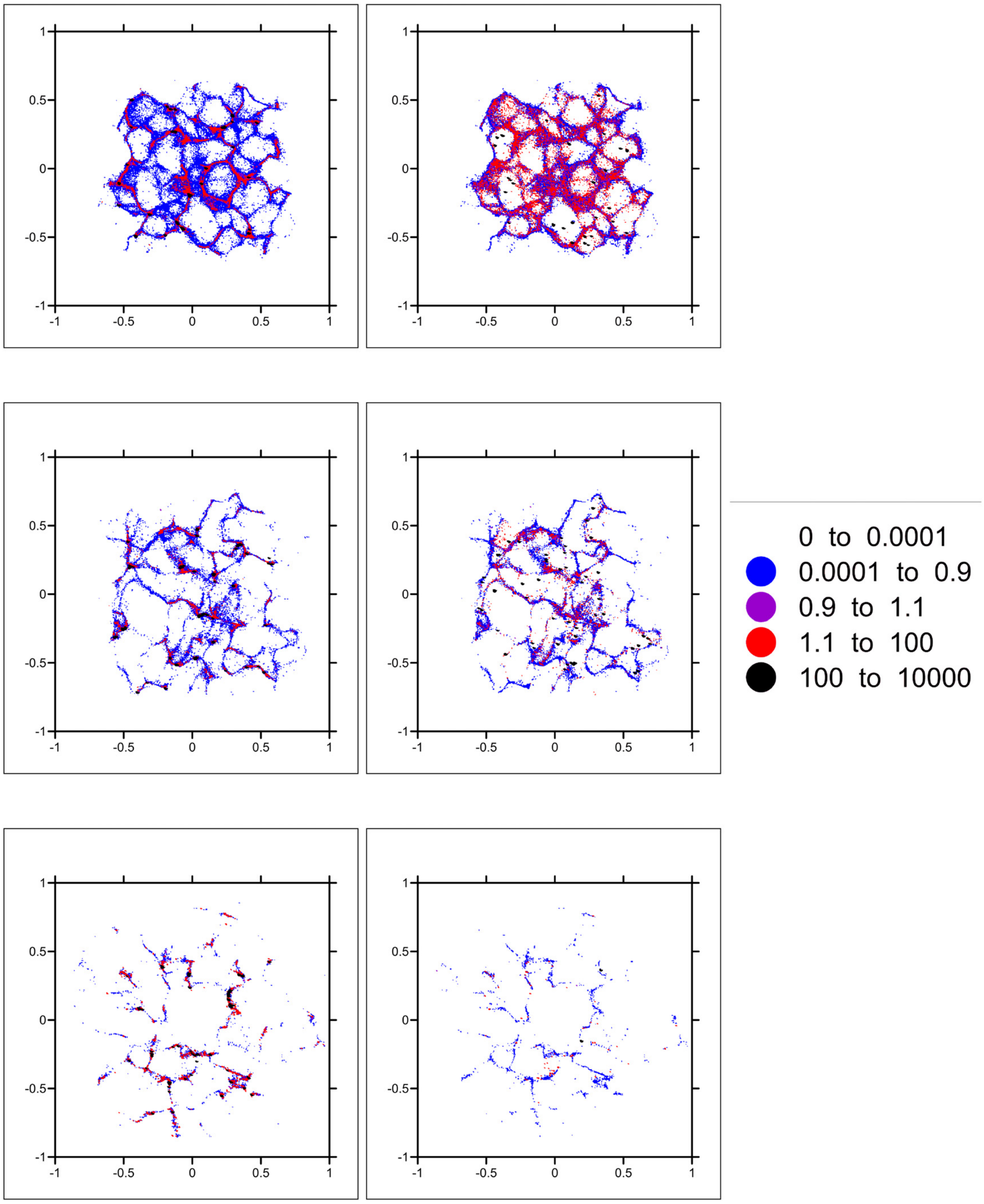

FIG. 8. Distribution of the tracer density (right) and concentration of tracer particles (left). $\sigma_{\mathbf{u}} = 2$, $l = 0.04 = 8h$, $\frac{\lambda_p}{cl} = 0.1$. The distributions are given for the following times (from top to bottom): $Dt = 4.883, 9.776, 19.532$.

obtained in the previous papers[7,22] were confirmed by a numerical model. In particular, the diffusion coefficients obtained in these works coincided with the numerical estimates. The effects of transport of tracer particles and clustering of the density field were also confirmed. However, certain new features were also identified: the areas of concentration of tracer particles, i.e., the areas where the tracer was predominantly transported by the random field did not coincide with the areas in which clustering took place. As a result of this discrepancy, although the conditions for clustering were met, only a small portion of the tracer was clustered. Moreover, the maximum density in these

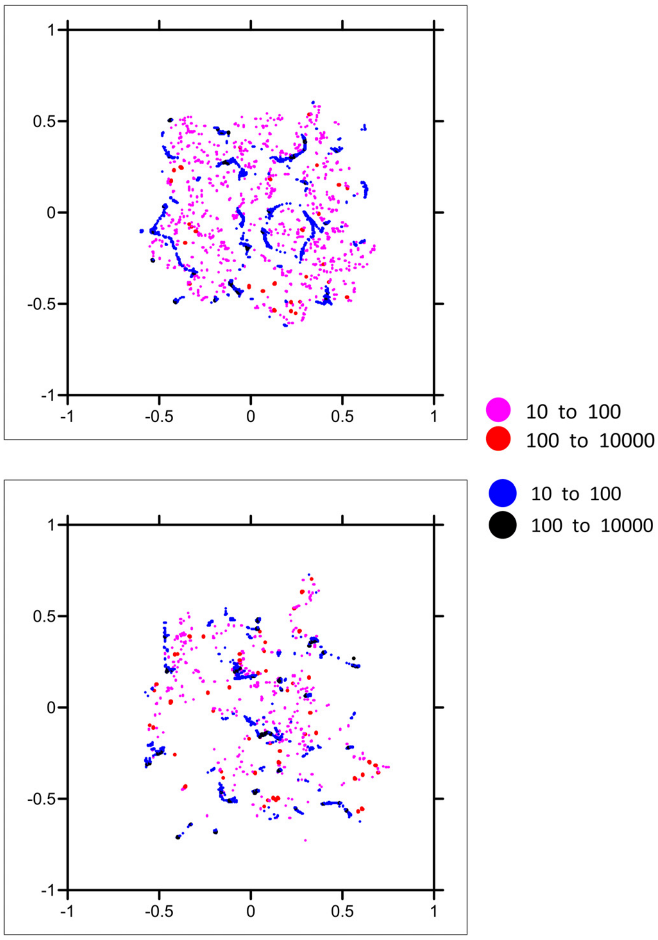

FIG. 9. Distribution of the tracer density (purple and red) and concentration of tracer particles (blue and black). $\sigma_{\mathbf{u}} = 2$, $l = 0.04 = 8h$, $\frac{\lambda_p}{cl} = 0.1$. The distributions are given for the following times (from top to bottom): $Dt = 9.776, 19.532$.

clusters did not increase exponentially, although it reached relatively high values. This is because the lifetime of clusters was shown to be limited, and exponential growth was associated with the merging of existing clusters. However, in this case, after the cluster decayed, a significant part of the tracer moved away from the region where clustering occurred.

Thus, we showed that clustering took place and depended on the parameters of the velocity field in accordance with the estimates from the works.[7,22] However, the effect was not global and only a portion of the tracer was susceptible to clustering with finite density values.

Another interesting result is that the transport of tracer particles in a random acoustic velocity field led to the concentration of tracer particles along certain lines, which can be called quasi-clusters, since the tracer density in these regions did not increase.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Konstantin V. Koshel: Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Dmitry V. Stepanov:** Formal analysis (equal); Investigation (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] A. Dolginov and N. Silant'ev, "Scalar field diffusion in a stochastic medium," J. Exp. Theor. Phys. **66**, 90–96 (1987).
[2] N. Silant'ev, "Comparison of methods for calculating turbulent diffusion coefficients," J. Exp. Theor. Phys. **84**, 479–484 (1997).
[3] T. Elperin, N. Kleeorin, and I. Rogachevskii, "Isotropic and anisotropic spectra of passive scalar fluctuations in turbulent fluid flow," Phys. Rev. E **53**, 3431–3441 (1996).
[4] T. Elperin, N. Kleeorin, and I. Rogachevskii, "Turbulent thermal diffusion of small inertial particles," Phys. Rev. Lett. **76**(2), 224–227 (1996).
[5] T. Elperin, N. Kleeorin, and I. Rogachevskii, "Self-excitation of fluctuations of inertial particle concentration in turbulent fluid flow," Phys. Rev. Lett. **77**(27), 5373–5376 (1996).
[6] L. Zaichik, V. M. Alipchenkov, and E. G. Sinaiski, *Particles in Turbulent Flows* (Wiley, 2008).
[7] V. Klyatskin and I. Yakushkin, "Stochastic transport in random wave fields," J. Exp. Theor. Phys. **91**, 736–747 (2000).
[8] A. Lichtenberg and M. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
[9] G. M. Zaslavsky, *The Physics of Chaos in Hamiltonian Systems* (Imperial College Press, London, 2007).
[10] A. Okubo, *Diffusion and Ecological Problems: Mathematical Models*, Biomathematics Vol. 10 (Springer-Verlag, Berlin, 1980).
[11] W. D. McComb, *The Physics of Fluid Turbulence*, Oxford Engineering Science Series Vol. 25 (Clarendon Press, Oxford, 1990).
[12] J. A. Jacobs, H. S. Huntley, A. D. Kirwan, B. L. Lipphardt, T. Campbell, T. Smith, K. Edwards, and B. Bartes, "Ocean processes underlying surface clustering," JGR Oceans **121**, 180 (2016).
[13] A. Kostinski and R. Shaw, "Scale-dependent droplet clustering in turbulent clouds," J. Fluid Mech. **434**, 389 (2001).
[14] G. Dagan, "Theory of solute transport by groundwater," Annu. Rev. Fluid Mech. **19**, 183–213 (1987).
[15] I. de Pater and J. Lissauer, *Planetary Science* (Cambridge University Press, 2001).
[16] S. F. Shandarin and Y. B. Zeldovich, "The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium," Rev. Mod. Phys. **61**, 185 (1989).

[17]V. I. Klyatskin, *Stochastic Equations: Theory and Applications in Acoustics, Hydrodynamics, Magnetohydrodynamics, and Radiophysics* (Springer, 2015), Vol. 1, p. 2.

[18]V. I. Klyatskin and D. Gurarie, "Coherent phenomena in stochastic dynamical systems," Phys.-Usp. **42**, 165 (1999).

[19]A. Saichev and W. Woyczynski, "Probability distributions of passive tracers in randomly moving media," in *Stochastic Models in Geosystems*, IMA Volumes in Mathematics and its Applications Vol. 85, edited by S. Molchanov and W. Woyczynski (Springer-Verlag, New York, 1996), pp. 359–400.

[20]C. L. Zirbel and E. Cinlar, "Mass transport by Brownian flows," in *Stochastic Models in Geosystems*, IMA Volumes in Mathematics and its Applications, edited by S. A. Molchanov and W. A. Woyczynski (Springer-Verlag, 1997), p. 459.

[21]V. I. Klyatskin and A. I. Saichev, "Statistical theory of the diffusion of a passive tracer in a random velocity field," J. Exp. Theor. Phys. **84**, 716–724 (1997).

[22]V. I. Klyatskin, "Stochastic dynamo in critical situations," Teor. Mat. Fiz. **172**, 415–436 (2012).

[23]K. V. Koshel and O. V. Alexandrova, "Some results of a numerical modeling of the diffusion of passive tracers in a random field of velocities," Izv. Atmos. Ocean. Phys. **35**, 578–588 (1999).

[24]V. Klyatskin and K. Koshel, "Impact of diffusion on surface clustering in random hydrodynamic flows," Phys. Rev. E **95**, 013109 (2017).

[25]K. V. Koshel, D. V. Stepanov, E. A. Ryzhov, P. Berloff, and V. I. Klyatskin, "Clustering of floating tracers in weakly divergent velocity fields," Phys. Rev. E **100**, 063108 (2019).

[26]J. R. Cressman, J. Davoudi, W. I. Goldburg, and J. Schumacher, "Eulerian and Lagrangian studies in surface flow turbulence," New J. Phys. **6**, 53 (2004).

[27]J. Larkin, W. Goldburg, and M. M. Bandi, "Time evolution of a fractal distribution: Particle concentrations in free-surface turbulence," Phys. D **239**, 1264–1268 (2010).

[28]S. Lovecchio, C. Marchioli, and A. Soldati, "Time persistence of floating-particle clusters in free-surface turbulence," Phys. Rev. Lett. **88**, 033003 (2013).

[29]V. I. Klyatskin, "Clustering and diffusion of particles and passive tracer density in random hydrodynamic flows," Phys.-Usp. **46**, 667–688 (2003).

[30]V. I. Klyatskin, "Statistical description of the diffusion of a passive tracer in a random velocity field," Phys.-Usp. **37**, 501–513 (1994).

[31]V. I. Klyatskin, "Stochastic structure formation in random media," Phys.-Usp. **59**, 67–95 (2016).

[32]P. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, Berlin, 1992).

[33]V. I. Klyatskin and K. V. Koshel, "The simplest example of the development of a cluster-structured passive tracer field in random flows," Usp. Fiz. Nauk **170**, 771–778 (2000).