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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

Features of Rayleigh Scattering by a Particle Near an Interface

A. O. Maksimov*

Il'ichev Pacific Oceanological Institute, Far East Branch, Russian Academy of Sciences, Vladivostok, 690041 Russia

**e-mail: maksimov@poi.dvo.ru*

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Abstract—Features of Rayleigh scattering by a solid particle at a small distance compared to the wavelength from an impenetrable plane boundary are revealed. The choice of the Green's function in the integral representation of the Helmholtz equation makes it possible to reduce integration only over the particle surface and eliminate the contribution of the interface surface. When expanding over a small wave parameter, a well-known approach is used, making it possible to represent the solution of a given order as the sum of a potential function and a component expressed in terms of lower-order approximations. The potential component is found, expressed in terms of solid irregular harmonics centered on the particle and its mirror image. The vibrational velocity of the center of a particle and the scattering amplitude are determined. In the lowest order of the wavenumber, the scattering amplitude is expressed in terms of the monopole and dipole components.

Keywords: Rayleigh scattering, hard sphere, plane boundary, vibrational velocity, scattering amplitude

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INTRODUCTION

This study is a continuation of [1, 2], in which the authors obtained an analytical description of the dynamics of a gas inclusion at a small distance from the interface between two contacting media. In contrast to a bubble, the inertial properties of the particle material cannot be neglected, and the approach used in [1, 2] requires modification. Sound scattering by an object at a short distance from a boundary has been a subject of research for several decades. Interest in this problem, stemming from hydroacoustics problems, involves scattering of high-frequency signals [3–6]. At the same time, with the advancement of acoustic methods for manipulating objects in biofabrication, acoustofluidics, and ultrasonic cleaning problems [7–10], as a rule, the sizes of objects are small compared to the wavelength.

This study is aimed at developing acoustic methods for manipulating small objects in the presence of bounding surfaces, in particular, finding the radiation pressure force under such conditions. The study is based on recent results [11–14]. The necessary first step is to find a first approximation, i.e., solve the linear problem of scattering by a particle at a small distance from the interface. The emphasis is on obtaining an approximate analytical description that offers a visual interpretation of the results and using analytical expressions in analyzing nonlinear effects.

FORMULATION OF THE PROBLEM

An incident wave φ_{in} is scattered by a target consisting of a hard particle with a surface S_{p}, located above the lower half-space with an impenetrable boundary S_{g} $(z = 0)$. The particle radius R_{p} and distance from the particle center to the boundary h are much smaller than the wavelength. The geometry of the problem is illustrated in Fig. 1.

Let us introduce notation for the point $\mathbf{r} = (x, y, z)$ and its mirror image $\mathbf{r}_i = (x, y, -z)$. We consider the function φ_0, which describes the solution to the problem of scattering by the boundary in the absence of a particle $\varphi_0(\mathbf{r}) = \varphi_{\text{in}}(\mathbf{r}) + \varphi_{\text{in}}(\mathbf{r}_i)$.

The scattered field φ^{s} satisfies the Helmholtz equation and the following boundary conditions:

$$\nabla^2\varphi^{\text{s}}(\mathbf{r}) + k^2\varphi^{\text{s}}(\mathbf{r}) = 0, \quad (\mathbf{n}\nabla)\varphi^{\text{s}} + (\mathbf{n}\nabla)\varphi_0 = (\mathbf{nu}),$$
$$\mathbf{r} \in S_{\text{p}}, \quad (\mathbf{n}\nabla)\varphi^{\text{s}} = 0, \quad \mathbf{r} \in S_{\text{g}}. \tag{1}$$

Here $\mathbf{u}$ is the velocity of the particle's center of mass, and $\mathbf{n}$ is the external normal with respect to both the particle and the lower medium.

The Helmholtz equation can be written in integral form:

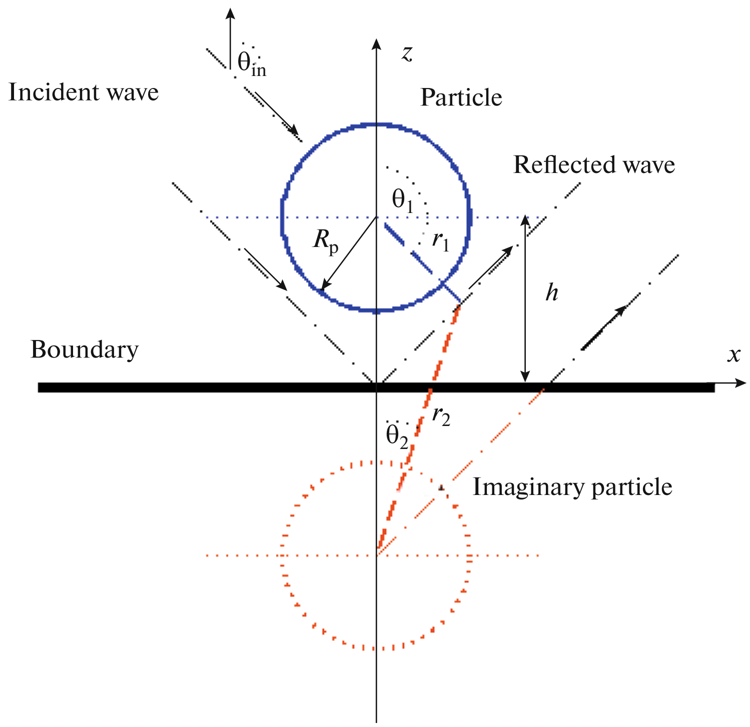

Fig. 1. Geometry of problem: incident wave propagates in direction of wave vector $\mathbf{k}/k = (\sin\theta_{in}, 0, \cos\theta_{in})$ and is scattered by particle with radius R_p, located at distance h from boundary of impermeable medium. A mirror particle is used to describe interaction with boundary. Spherical coordinate systems centered on particle and its mirror image $\mathbf{r}_1 = (r_1, \theta_1, \alpha)$, $\mathbf{r}_2 = (r_2, \theta_2, \alpha), \mathbf{r}_2 = \mathbf{r}_1 + 2h\mathbf{e}_z$, are used in constructing potential solution to boundary value problem.

$$\varphi^s(\mathbf{r}) = \frac{1}{4\pi} \times \int\limits_{S_p+S_g} \left\{ \varphi^s(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi^s(\mathbf{r}')}{\partial n'} \right\} dS', \tag{2}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the Green's function. Selecting a Green's function that satisfies the boundary condition on S_g $(z = 0)$,

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{ikR}}{R} + \frac{e^{ikR_i}}{R_i},$$
$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$
$$R_i = \sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}$$

leads to the fact that integration in formula (2) is done only over the particle surface.

Because $\varphi_0(\mathbf{r})$ also satisfies this integral equation, for the total field $\varphi^T(\mathbf{r}) = \varphi_0(\mathbf{r}) + \varphi^s(\mathbf{r})$, we have

$$\varphi^T(\mathbf{r}) = \varphi_0(\mathbf{r}) + \frac{1}{4\pi} \times \int\limits_{S_p} \left\{ \varphi^T(\mathbf{r}') \frac{\partial}{\partial n'} \left[\frac{e^{ikR}}{R} + \frac{e^{ikR_i}}{R_i} \right] - \left[\frac{e^{ikR}}{R} + \frac{e^{ikR_i}}{R_i} \right] \frac{\partial \varphi^T(\mathbf{r}')}{\partial n'} \right\} dS'. \tag{3}$$

When finding the solution in the long-wavelength approximation, we use the technique proposed in [15]. Let us present the low-frequency expansion of the quantities entering into formula (3) as

$$\varphi_0(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ikR_p)^n}{n!} \varphi_{0n}(\mathbf{r}),$$
$$\varphi^T(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ikR_p)^n}{n!} \varphi_n^T(\mathbf{r}), \tag{4}$$
$$\mathbf{u}(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ikR_p)^n}{n!} \mathbf{u}_n(\mathbf{r}).$$

If $\varphi_{in}(\mathbf{r})$ is a plane wave $\varphi_{in}(\mathbf{r})$ = $\varphi_m \exp[i(\mathbf{kr}) - i\omega t]$ propagating in the direction $\mathbf{e}_k = \mathbf{k}/k$, then $\varphi_{0n}(\mathbf{r})/\varphi_m = \left[(\mathbf{e}_k\mathbf{r})R_p^{-1}\right]^n + \left[(\mathbf{e}_k\mathbf{r}_i)R_p^{-1}\right]^n$. In a spherical coordinate system associated with the center of the particle

$$\mathbf{e}_k = (\sin\theta_{in}\cos\alpha_{in}, \sin\theta_{in}\sin\alpha_{in}, \cos\theta_{in}).$$

Substituting the expansion of the Green's function, we obtain equations for the sought quantities:

$$\varphi_n^T(\mathbf{r}) = \varphi_{0n}(\mathbf{r}) + \frac{1}{4\pi}\sum_{l=0}^{n}\binom{n}{l}\frac{1}{R_p^l} \times \int_{S_p}\left\{\varphi_{n-l}^T(\mathbf{r}')\frac{\partial}{\partial n'}\left[R^{l-1} + R_i^{l-1}\right] - \left[R^{l-1} + R_i^{l-1}\right](\mathbf{u}_{n-l}\mathbf{n}')\right\}dS'. \tag{5}$$

This equation expresses the nth term of the expansion in all previous ones, including the nth. However, the advantage of this notation is that the term on the right-hand side containing φ_n^{T}, is a potential function: a solution to the Laplace equation [15]. Therefore, the sought solution can be represented as

$$\begin{gathered}\varphi_n^{\mathrm{T}}(\mathbf{r}) = F_n(\mathbf{r}) + \phi_n(\mathbf{r}),\\ F_n(\mathbf{r}) = \varphi_{0n}(\mathbf{r}) + \frac{1}{4\pi}\sum_{l=1}^{n}\binom{n}{l}\frac{1}{R_p^l} \times \int_{S_\mathrm{p}}\left\{\varphi_{n-l}^{\mathrm{T}}(\mathbf{r}')\frac{\partial}{\partial n'}\left[R^{l-1} + R_i^{l-1}\right] - \left[R^{l-1} + R_i^{l-1}\right]\right\}(\mathbf{u}_{n-l}\mathbf{n})\,dS',\\ \phi_n(\mathbf{r}) = \frac{1}{4\pi}\int_{S_\mathrm{p}}\varphi_n^{\mathrm{T}}(\mathbf{r}')\frac{\partial}{\partial n'}\left[\frac{1}{R} + \frac{1}{R_i}\right]dS',\\ \binom{n}{l} = \frac{n!}{l!(n-l)!}.\end{gathered} \tag{6}$$

Thus, if we know $\varphi_m^{\mathrm{T}}(\mathbf{r})$ for $m = 0.1, \ldots(n-1)$, then finding $\varphi_n^{\mathrm{T}}(\mathbf{r})$ reduces to solving the following boundary value problem for ϕ_n:

$$\begin{gathered}\nabla^2\phi_n(\mathbf{r}) = 0,\quad \frac{\partial\phi_n}{\partial n'} + \frac{\partial F_n}{\partial n'} = (\mathbf{u}_n\mathbf{n}),\\ \mathbf{r}\in S_\mathrm{p},\quad \frac{\partial\phi_n}{\partial n'} = 0,\quad \mathbf{r}\in S_\mathrm{g}.\end{gathered} \tag{7}$$

The first terms of the expansion are as follows:

$$\begin{gathered}F_0 = \varphi_{00} = 2\varphi_m,\quad \phi_0(\mathbf{r}) = 0,\quad \mathbf{u}_0 = 0,\\ \varphi_0^T = F_0 + \phi_0 = 2\varphi_m,\quad F_1 = \varphi_{01} = 2\varphi_m\\ \times[\sin\theta\sin\theta_\mathrm{in}\cos(\alpha-\alpha_\mathrm{in})(r/R_\mathrm{p}) + \cos\theta\cos\theta_\mathrm{in}(h/R_\mathrm{p})].\end{gathered} \tag{8}$$

The description of Rayleigh scattering by a solid particle near the interface is thus reduced to solving the following boundary value problem:

$$\begin{gathered}\nabla^2\phi_1(\mathbf{r}) = 0,\quad \left.\frac{\partial\phi_1}{\partial r}\right|_{r=R_p}\\ = -2\left(\frac{\varphi_m}{R_p}\right)\sin\theta\sin\theta_\mathrm{in}\cos(\alpha-\alpha_\mathrm{in}) + (\mathbf{u}_1\mathbf{n}),\\ \left.\frac{\partial\phi_1}{\partial z}\right|_{z=0} = 0.\end{gathered} \tag{9}$$

The direction of the x axis can be chosen along the projection of the wave vector, so that $\alpha_\mathrm{in} = 0$.

CONSTRUCTING THE SOLUTION

Following [16], we seek a solution in the form of the sum of the potentials centered on the particle and its mirror image:

$$\begin{gathered}\phi_1 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{lm}R_p^{l+1}I_{lm}(\mathbf{r}_1) + b_{lm}R_\mathrm{p}^{l+1}I_{lm}(\mathbf{r}_2)\right),\\ I_{lm}(\mathbf{r}) = \sqrt{\frac{4\pi}{2l+1}}\frac{Y_{lm}(\theta,\alpha)}{r^{l+1}},\quad \mathbf{r}_2 = \mathbf{r}_1 + 2h\mathbf{e}_z.\end{gathered} \tag{10}$$

Here Y_{lm} are spherical functions, I_{lm} are irregular spatial harmonics. In order to satisfy the boundary conditions at $z = 0$, the following conditions must be met: $b_{lm} = (-1)^{1+m}a_{lm}$.

To satisfy boundary conditions on the particle surface, it is necessary to transform solid harmonics centered on the mirror image to coordinates centered on the particle. The addition theorem [17] provides this relationship:

$$I_{lm}(\mathbf{r}_1 + 2h\mathbf{e}_z) = \sum_{\lambda=|m|}^{\infty}(-1)^{\lambda+m}\sqrt{\frac{4\pi}{2\lambda+1}} \times \sqrt{\frac{(l+\lambda)!(l+\lambda)!}{(l+m)!(l-m)!(\lambda-m)!(\lambda+m)!}} \times \frac{r_1^{\lambda}}{(2h)^{l+\lambda+1}}Y_{\lambda m}(\theta,\alpha). \tag{11}$$

The first approximation potential takes the form

$$\begin{gathered}\phi_1(r,\theta,\alpha) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}\left[\left(\frac{R_p}{r_1}\right)^{l+1}Y_{lm}(\theta,\alpha) + \sum_{\lambda=|m|}^{\infty}(-1)^{\lambda+l}\sqrt{\frac{2l+1}{2\lambda+1}}\right.\\ \left.\times\sqrt{\frac{(l+\lambda)!(l+\lambda)!}{(l+m)!(l-m)!(\lambda-m)!(\lambda+m)!}} \times \frac{r_1^{\lambda}R_p^{l+1}}{(2h)^{l+\lambda+1}}Y_{\lambda m}(\theta,\alpha)\right],\end{gathered} \tag{12}$$

where we retained the notation a_{lm} for renormalized expansion coefficients $a_{lm}\sqrt{4\pi/(2l+1)} \to a_{lm}$. Substitution of (12) into kinematic boundary condition (9) and projecting it onto Y^*_{LM} yield

$$a_{LM} - \frac{L}{L+1}\sum_{l=|M|}^{\infty} a_{lM}(-1)^{l+L}\sqrt{\frac{2l+1}{2L+1}}\left(\frac{R_p}{2h}\right)^{l+L+1} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+M)!(l-M)!(L-M)!(L+M)!}} = -\frac{1}{L+1}\sqrt{\frac{2\pi}{3}}[\varphi_0\sin\theta_{in} - u_{1x}R_p]\delta_{L1}(\delta_{M1}-\delta_{M-1}). \quad (13)$$

The appearance of the right-hand side determines the presence only of $M = \pm 1$.

The power-law dependence of the coefficients on parameter $\varepsilon = (R_p/2h)$ makes it possible, following [16], to seek a solution in the form

$$a_{L1} = \frac{1}{L+1}\sqrt{\frac{2\pi}{3}}[\sin\theta_{in}\varphi_0 - u_{1x}R_p] \times \left(\frac{R_p}{2h}\right)^{L-1}\sum_{k=0}^{\infty}\alpha_{Lk}\left(\frac{R_p}{2h}\right)^k,$$

$$\sum_{k=0}^{\infty}\alpha_{Lk}\left(\frac{R_p}{2h}\right)^k - L\sum_{l=1}^{\infty}(-1)^{l+L}\frac{1}{l+1}\sqrt{\frac{2l+1}{2L+1}} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+1)!(l-1)!(L-1)!(L+1)!}} \times \sum_{m=0}^{\infty}\alpha_{lm}\left(\frac{R_p}{2h}\right)^{2l+1+m} = -\delta_{L1}. \quad (14)$$

Equating coefficients with the same powers ε, we obtain

$$\alpha_{Lk} - L\sum_{l=1}^{\infty}(-1)^{l+L}\frac{1}{(l+1)}\sqrt{\frac{2l+1}{2L+1}} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+1)!(l-1)!(L-1)!(L+1)!}}\alpha_{l(k-2l-1)} = -\delta_{L1}\delta_{k0}. \quad (15)$$

As noted in [16], the resulting relations are essentially recurrent formulas that make it possible to find the solution in explicit form:

$$\alpha_{10} = -1,\ \alpha_{L0} = 0\ (L = 2,3,\ldots),\ \alpha_{L1} = 0,\ \alpha_{L2} = 0,$$

$$\alpha_{L3} = L\sum_{l=1}^{\infty}(-1)^{l+L}\frac{1}{(l+1)}\sqrt{\frac{2l+1}{2L+1}} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+1)!(l-1)!(L-1)!(L+1)!}}\alpha_{l(3-2l-1)} = L(-1)^{1+L}\frac{1}{2}\sqrt{\frac{3}{2L+1}}\sqrt{\frac{(1+L)!}{2(L-1)!}}\alpha_{10} = L(-1)^L\frac{1}{2}\sqrt{\frac{3}{2L+1}}\sqrt{\frac{(1+L)!}{2(L-1)!}},$$

$$\alpha_{13} = -(1/2),\ \alpha_{23} = \frac{3}{\sqrt{5}},\ \alpha_{L4} = 0,\ \alpha_{L5} = 0,$$

$$\alpha_{L6} = L\sum_{l=1}^{\infty}(-1)^{l+L}\frac{1}{(l+1)}\sqrt{\frac{2l+1}{2L+1}} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+1)!(l-1)!(L-1)!(L+1)!}}\alpha_{l(6-2l-1)} = L(-1)^{1+L}\frac{1}{2}\sqrt{\frac{3}{2L+1}}\sqrt{\frac{(1+L)!}{2(L-1)!}}\alpha_{13} = -\alpha_{L3}\alpha_{13} = \alpha_{L3}/2,\ \alpha_{L7} = 0, \quad (16)$$

$$\alpha_{L8} = L\sum_{l=1}^{\infty}(-1)^{l+L}\frac{1}{(l+1)}\sqrt{\frac{2l+1}{2L+1}} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+1)!(l-1)!(L-1)!(L+1)!}}\alpha_{l(8-2l-1)} = (-1)^L\frac{L(2+L)}{3}\sqrt{\frac{5}{2L+1}}\sqrt{\frac{(1+L)!}{3!(L-1)!}}\alpha_{23} = (-1)^L L(2+L)\sqrt{\frac{(1+L)!}{3!(2L+1)(L-1)!}},$$

$$\alpha_{L9} = L\sum_{l=1}^{\infty}(-1)^{l+L}\frac{1}{(l+1)}\sqrt{\frac{2l+1}{2L+1}} \times \sqrt{\frac{(l+L)!(l+L)!}{(l+1)!(l-1)!(L-1)!(L+1)!}}\alpha_{l(9-2l-1)} = L(-1)^{1+L}\frac{1}{2}\sqrt{\frac{3}{2L+1}}\sqrt{\frac{(1+L)!}{2(L-1)!}}\alpha_{16} = -\alpha_{L3}\alpha_{16} = \alpha_{L3}/4.$$

The accuracy in representing the solution is determined by the power of parameter $\varepsilon = (R_p/2h) > (1/2)$. Thus, an accuracy of two orders of magnitude is ensured by taking into account terms up to $\varepsilon^7 \approx 0.01$. With this accuracy, the solution has the form

$$\phi_1(r,\theta,\alpha) = \sum_{l=1}^{5} a_{l1}\left[\left(\frac{R_p}{r_1}\right)^{l+1}\left(Y_{l1}(\theta_1,\alpha) + Y^*_{l1}(\theta_1,\alpha)\right) + (-1)^{l+1}\left(\frac{R_p}{r_2}\right)^{l+1}\left(Y_{l1}(\theta_2,\alpha) + Y^*_{l1}(\theta_2,\alpha)\right)\right],$$

$$a_{11} = -\frac{1}{2}\sqrt{\frac{2\pi}{3}}[-2\sin\theta_{in}\varphi_m + u_{1x}R_p] \times \left(\alpha_{10} + \alpha_{13}\left(\frac{R_p}{2h}\right)^3 + \alpha_{16}\left(\frac{R_p}{2h}\right)^6\right) = -\frac{1}{2}\sqrt{\frac{2\pi}{3}}[2\sin\theta_{in}\varphi_m - u_{1x}R_p] \times \left(1 + \frac{1}{2}\left(\frac{R_p}{2h}\right)^3 + \frac{1}{4}\left(\frac{R_p}{2h}\right)^6\right),$$

$$a_{21} = -\frac{1}{3}\sqrt{\frac{2\pi}{3}}[-2\sin\theta_{\text{in}}\varphi_m + u_{1x}R_p] \times \left(\alpha_{23}\left(\frac{R_p}{2h}\right)^4 + \alpha_{26}\left(\frac{R_p}{2h}\right)^7\right) = \sqrt{\frac{2\pi}{3\times 5}}[2\sin\theta_{\text{in}}\varphi_m - u_{1x}R_p]\left(\left(\frac{R_p}{2h}\right)^4 + \frac{1}{2}\left(\frac{R_p}{2h}\right)^7\right), \quad (17)$$

$$a_{31} = -\frac{1}{4}\sqrt{\frac{2\pi}{3}}[-2\sin\theta_{\text{in}}\varphi_m + u_{1x}R_p] \times \left(\alpha_{33}\left(\frac{R_p}{2h}\right)^5 + \alpha_{36}\left(\frac{R_p}{2h}\right)^8\right) = \frac{9}{8}\sqrt{\frac{2\pi}{3}}[2\sin\theta_{\text{in}}\varphi_m - u_{1x}R_p] \times \left(-\sqrt{\frac{2}{7}}\left(\frac{R_p}{2h}\right)^5 - \sqrt{\frac{7}{13}}\left(\frac{R_p}{2h}\right)^8\right),$$

$$a_{41} = -\frac{1}{5}\sqrt{\frac{2\pi}{3}}[-2\sin\theta_{\text{in}}\varphi_m + u_{1x}R_{\text{p}}]\alpha_{43}\left(\frac{R_{\text{p}}}{2h}\right)^6 = \frac{4}{3}\sqrt{\frac{\pi}{5}}[2\sin\theta_{\text{in}}\varphi_m - u_{1x}R_p]\left(\frac{R_{\text{p}}}{2h}\right)^6,$$

$$a_{51} = -\frac{1}{6}\sqrt{\frac{2\pi}{3}}[-2\sin\theta_{\text{in}}\varphi_m + u_{1x}R_{\text{p}}]\alpha_{53}\left(\frac{R_{\text{p}}}{2h}\right)^7 = -\frac{5}{4}\sqrt{\frac{2\pi\times 5}{3\times 11}}[2\sin\theta_{\text{in}}\varphi_m - u_{1x}R_{\text{p}}]\left(\frac{R_{\text{p}}}{2h}\right)^7.$$

The next step is to find the vibrational velocity of motion of the center of the particle.

PARTICLE VIBRATIONAL VELOCITY

The velocity of the center of the particle is determined from the condition of the balance of inertial forces and pressure acting on the surface of the particle, which in the approximation linear in the wavenumber is reduced to

$$\rho_{\text{p}}\frac{4\pi R_p^3}{3}\frac{d\mathbf{u}}{dt} = -\int_{S_{\text{p}}} p\mathbf{n}'\,dS', \qquad \rho_{\text{p}}\frac{4\pi R_{\text{p}}^3}{3}\mathbf{u}_1 = \rho_w\int_{S_{\text{p}}}\left(F_1(\mathbf{r}') + \phi_1(\mathbf{r}')\right)\mathbf{n}'\,dS', \quad (18)$$

where ρ_{p} is the particle density and ρ_w is the density of liquid. In the linear approximation, a particle can only oscillate parallel to the boundary surface, along the direction of the incident wave. The condition of impermeability of the boundary leads to vanishing of the linear (for parameter $kR_{\text{p}} \ll 1$) normal velocity component. The normal component appears when taking into account the next order of perturbation theory for parameter ($kR_{\text{p}} \ll 1$). Calculating the surface integral results in the following expression:

$$\rho_{\text{p}}\frac{4\pi R_{\text{p}}^3}{3}u_{1x} = \rho_w R_{\text{p}}^2\int_0^{\pi}\int_0^{2\pi}\sin\theta'\,d\theta'\,d\alpha' \times \left(F_1(\theta',\alpha') + \phi_1(\theta',\alpha')\right)\sin\theta'\cos\alpha',$$

$$\rho_w R_{\text{p}}^2\int_0^{\pi}\int_0^{2\pi}\sin\theta'\,d\theta'\,d\alpha' F_1(\theta',\alpha')\sin\theta'\cos\alpha' = \rho_w\frac{4\pi}{3}R_{\text{p}}^2\varphi_m 2\sin\theta_{\text{in}},$$

$$\rho_w R_{\text{p}}^2\int_0^{\pi}\int_0^{2\pi}\sin\theta'\,d\theta'\,d\alpha'\phi_1(\theta',\alpha')\sin\theta'\cos\alpha' = \rho_w R_{\text{p}}^2\sqrt{\frac{8\pi}{3}}\left\{a_{11}\left(1 + \left(\frac{R_{\text{p}}}{2h}\right)^3\right) + \sum_{l=2}^{\infty}a_{l1}(-1)^{l+1}\sqrt{\frac{2l+1}{3}}\sqrt{\frac{(l+1)!}{2(l-1)!}}\left(\frac{R_{\text{p}}}{2h}\right)^{l+2}\right\}, \quad (19)$$

$$\rho_{\text{p}}u_{1x} = \rho_w\left\{\varphi_m R_{\text{p}}^{-1}2\sin\theta_{\text{in}} - \left[\varphi_m R_{\text{p}}^{-1}2\sin\theta_{\text{in}} - u_{1x}\right] \times \left(-\frac{1}{2} + M\right)\right\},$$

$$M = \frac{1}{2} + \frac{1}{2}\left[1 + \left(\frac{R_{\text{p}}}{2h}\right)^3\right]\sum_{k=0}^{\infty}\alpha_{1k}\left(\frac{R_{\text{p}}}{2h}\right)^k + \sum_{l=2}^{\infty}\frac{(-1)^{l+1}}{l+1}\sqrt{\frac{2l+1}{3}}\sqrt{\frac{(l+1)!}{2(l-1)!}}\left(\frac{R_{\text{p}}}{2h}\right)^{2l+1}\sum_{k=0}^{\infty}\alpha_{lk}\left(\frac{R_{\text{p}}}{2h}\right)^k \approx -\frac{3}{4}\left(\frac{R_{\text{p}}}{2h}\right)^3 - \frac{3}{8}\left(\frac{R_{\text{p}}}{2h}\right)^6,$$

$$u_{1x} = \varphi_m R_{\text{p}}^{-1}2\sin\theta_{\text{in}}\frac{\rho_w(3-2M)}{2\rho_{\text{p}} + \rho_w(1-2M)},$$

$$u_x = v_x^{(0)}\frac{\rho_w(3-2M)}{2\rho_{\text{p}} + \rho_w(1-2M)}.$$

Here, $v_x^{(0)} = 2ik_x\varphi_m$ is the velocity induced by waves incident and reflected from the interface at a point coinciding with the center of the particle, but in the absence of the particle itself. As the distance to the boundary increases $M \to 0$, the expression for velocity transforms into the well-known formula for a free particle.

FAR FIELD—SCATTERING AMPLITUDE

The behavior of the scattered field in the far field $kr \gg 1$ can conveniently be described in terms of the scattering amplitude: $\varphi^s/\varphi_0 \approx f(\theta,\alpha)\left(e^{ikr}/r\right)$. Using the representation for scattered field (2) and the asymptotics for the Green's function $\left(e^{ikR}/R\right)+\left(e^{ikR_i}/R_i\right) \approx \left(e^{ikr}/r\right)\left[e^{-ik(\mathbf{e}_r\mathbf{r}')}+e^{-ik(\mathbf{e}_r\mathbf{r}_i')}\right]$, we obtain

$$f(\theta,\alpha)=\frac{1}{4\pi\varphi_0}\int\limits_{S_\mathrm{p}}\left\{\varphi^\mathrm{T}(\mathbf{r}')\frac{\partial}{\partial n'}\left[e^{-ik(\mathbf{e}_r\mathbf{r}')}+e^{-ik(\mathbf{e}_r\mathbf{r}_i')}\right]\right.\\ \left.-\left[e^{-ik(\mathbf{e}_r\mathbf{r}')}+e^{-ik(\mathbf{e}_r\mathbf{r}_i')}\right](\mathbf{u}\mathbf{n}')\right\}dS'. \tag{20}$$

Substituting into this expression the low-frequency expansion for potential (4) and the components of the Green's function, we obtain

$$f(\theta,\alpha)=\sum_{l=1}^{\infty}\frac{(ikR_\mathrm{p})^l}{l!}\sum_{m=1}^{l}(-1)^m\frac{l!}{(l-m)!\,m!}\frac{1}{4\pi\varphi_0}\\ \times\int\limits_{S_p}\left\{-\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})^m+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})^m\right]\right.\\ \times(\mathbf{u}_{l-m}\mathbf{n}')+\frac{\partial}{\partial n'}\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})^m+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})^m\right]\\ \left.\times\left[F_{l-m}(\mathbf{r}')+\phi_{l-m}(\mathbf{r}')\right]\right\}dS'. \tag{21}$$

The first term of expansion $l = 1$, $m = 1$ makes no contribution. For second-order terms ($l = 2$), we have

$$f_2(\theta,\alpha)=-\frac{(kR_\mathrm{p})^2}{4\pi\varphi_0}\int\limits_{S_\mathrm{p}}\left\{\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})\right](\mathbf{u}_1\mathbf{n}')\right.\\ -\frac{\partial}{\partial n'}\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})\right]\times\\ \times\left[F_1(\mathbf{r}')+\phi_1(\mathbf{r}')\right]+\frac{1}{2}\frac{\partial}{\partial n'}\\ \left.\times\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})^2+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})^2\right]F_0(\mathbf{r}')\right\}dS'. \tag{22}$$

Calculation of the individual terms entering into this expression leads to the following results:

$$f_2^{(1)}(\theta,\alpha)=-\frac{(kR_\mathrm{p})^2}{4\pi\varphi_0}\\ \times\int\limits_{S_\mathrm{p}}dS'\frac{1}{2}\frac{\partial}{\partial n'}\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})^2+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})^2\right]F_0(\mathbf{r}')\\ =-\frac{k^2R_\mathrm{p}^3}{4\pi}\int\limits_0^\pi\sin\theta'\,d\theta'\int\limits_0^{2\pi}d\alpha'(\mathbf{e}_r^2\mathbf{n}')^2=-k^2R_\mathrm{p}^3\frac{2}{3}. \tag{23}$$

This term is associated with a monopole source at the center of the particle, which takes into account the compressibility of the medium surrounding the particle (in the second-order for the wave number) caused by the incident field. When describing this contribution, we can take into account the compressibility of the particle material, which will lead to the appearance of the factor $\left(1-c_w^2\rho_w/c_\mathrm{p}^2\rho_\mathrm{p}\right)$.

Calculating the contribution of the term associated with the vibrational velocity of the particle gives

$$f_2^{(2)}(\theta,\alpha)=-\frac{(kR_\mathrm{p})^2}{4\pi\varphi_0}\\ \times\int\limits_{S_p}\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})\right](\mathbf{u}_1\mathbf{n}')\,dS'\\ =-\frac{k^2R_\mathrm{p}^4}{2\pi\varphi_0}\int\limits_0^\pi\sin\theta'\,d\theta'\int\limits_0^{2\pi}d\alpha'(\mathbf{e}_r\mathbf{n}')(\mathbf{u}_1\mathbf{n}')\\ =-k^2R_\mathrm{p}^3\frac{2R_\mathrm{p}}{3\varphi_0}(\mathbf{e}_r\mathbf{u}_1). \tag{24}$$

This contribution is due to the dipole source at the center of the particle, which takes into account vibrational displacements of the center. However, since the relative velocity (relative to the medium) is important for radiation, the sum with the term below has physical meaning:

$$f_2^{(3a)}(\theta,\alpha)=\frac{(kR_\mathrm{p})^2}{4\pi\varphi_0}\\ \times\int\limits_{S_\mathrm{p}}\frac{\partial}{\partial n'}\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p})+(\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})\right]F_1(\mathbf{r}')dS'\\ =\frac{k^2R_\mathrm{p}^3}{2\pi}\int\limits_0^\pi\sin\theta'\,d\theta'\int\limits_0^{2\pi}d\alpha'\sin\theta'\sin\theta_\mathrm{in}\cos(\alpha')(\mathbf{e}_r\mathbf{n}')\\ =k^2R_\mathrm{p}^3\frac{(\mathbf{e}_r^2\mathbf{e}_{k_\perp})}{3}. \tag{25}$$

Here, $\mathbf{e}_{k_\perp}=\mathbf{k}_\perp/k=(\sin\theta_\mathrm{in},0,0)$. This term is associated with a dipole source at the center of the particle, which takes into account the vibrational displacements of the medium caused by the incident wave.

Calculation of the contribution associated with the scattered field is the most cumbersome:

$$f_2^{(3b)}(\theta,\alpha) = \frac{(kR_\mathrm{p})^2}{4\pi\varphi_0}\int_{S_\mathrm{p}}\frac{\partial}{\partial n'}\left[(\mathbf{e}_r\mathbf{r}'/R_\mathrm{p}) + (\mathbf{e}_r\mathbf{r}_i'/R_\mathrm{p})\right]\phi_1(\mathbf{r}')dS'$$

$$= \frac{k^2R_\mathrm{p}^3}{4\pi\varphi_0}\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\alpha' 2\left[\sin\theta\sin\theta'\cos(\alpha-\alpha') + \cos\theta\cos\theta' + (h/R_\mathrm{p})\cos\theta'\right]$$

$$\times\left\{\sum_{l=1}^\infty a_{l1}\left[Y_{l1}(\theta',\alpha') + \sum_{\lambda=1}^\infty(-1)^{\lambda+l}\sqrt{\frac{2l+1}{2\lambda+1}}\sqrt{\frac{(l+\lambda)!(l+\lambda)!}{(l+1)!(l-1)!(\lambda-1)!(\lambda+1)!}}\frac{R_\mathrm{p}^{l+\lambda+1}}{(2h)^{l+\lambda+1}}Y_{\lambda 1}(\theta',\alpha')\right]\right.$$

$$\left.+\sum_{l=1}^\infty a_{l-1}\left[Y_{l-1}(\theta',\alpha') + \sum_{\lambda=1}^\infty(-1)^{\lambda+l}\sqrt{\frac{2l+1}{2\lambda+1}}\sqrt{\frac{(l+\lambda)!(l+\lambda)!}{(l+1)!(l-1)!(\lambda-1)!(\lambda+1)!}}\frac{R_\mathrm{p}^{l+\lambda+1}}{(2h)^{l+\lambda+1}}Y_{\lambda-1}(\theta',\alpha')\right]\right\} \quad (26)$$

$$= -k^2R_\mathrm{p}^3\frac{1}{\sqrt{4\pi}}\sqrt{\frac{2}{3}}2\sin\theta\cos\alpha\frac{1}{\varphi_0}\left\{a_{11} + \sum_{l=1}^\infty(-1)^{l+1}a_{l1}\sqrt{\frac{2l+1}{3}}\sqrt{\frac{(l+1)!}{(l-1)!2}}\frac{R_\mathrm{p}^{l+2}}{(2h)^{l+2}}\right\}.$$

$$= -k^2R_\mathrm{p}^3\frac{2}{3}\sin\theta\cos\alpha\left[\sin\theta_{in} - u_{1x}R_\mathrm{p}/\varphi_0\right]\left\{\frac{1}{2}\sum_{k=0}^\infty\alpha_{1k}\left(\frac{R_p}{2h}\right)^k + \sum_{l=1}^\infty(-1)^{l+1}\frac{1}{(l+1)}\sqrt{\frac{2l+1}{3}}\right.$$

$$\left.\times\sqrt{\frac{(l+1)!}{(l-1)!2}}\frac{R_\mathrm{p}^{2l+1}}{(2h)^{2l+1}}\sum_{k=0}^\infty\alpha_{lk}\left(\frac{R_\mathrm{p}}{2h}\right)^k\right\} = \frac{2}{3}\left[(\mathbf{e}_r\mathbf{e}_{k_\perp}) - \frac{R_\mathrm{p}}{\varphi_0}(\mathbf{e}_r\mathbf{u}_1)\right]\left(\frac{1}{2}-M\right).$$

When calculating the contribution $\phi_1(\mathbf{r}')$, we make use of the fact that

$$a_{l1} = -a_{l-1},\quad \sin\theta'\cos\alpha' = \sqrt{2\pi/3}\left[Y_{1-1}^*(\theta',\alpha') - Y_{11}^*(\theta',\alpha')\right],$$
$$\sin\theta'\sin\alpha' = i\sqrt{2\pi/3}\left[-Y_{1-1}^*(\theta',\alpha') - Y_{11}^*(\theta',\alpha')\right].$$

Summing the contributions of individual terms, we obtain

$$f_2(\theta,\alpha) = -k^2R_\mathrm{p}^3\frac{2}{3} - k^2R_\mathrm{p}^3\frac{2R_\mathrm{p}}{3\varphi_0}(\mathbf{e}_r\mathbf{u}_1) + k^2R_\mathrm{p}^3\frac{(\mathbf{e}_r^2\mathbf{e}_{k_\perp})}{3}$$
$$+ k^2R_\mathrm{p}^3\frac{2}{3}\left[(\mathbf{e}_r\mathbf{e}_{k_\perp}) - \frac{R_\mathrm{p}}{\varphi_0}(\mathbf{e}_r\mathbf{u}_1)\right]\left(\frac{1}{2}-M\right)$$
$$= k^2R_\mathrm{p}^3\left[-\frac{1}{3} + (\mathbf{e}_r\mathbf{e}_{k_\perp})\frac{(\rho_\mathrm{p}-\rho_w)[1-(2/3)M]}{2\rho_\mathrm{p}+\rho_w(1-2M)}\right], \quad (27)$$
$$M \approx -\frac{3}{4}\left(\frac{R_\mathrm{p}}{2h}\right)^3 - \frac{3}{8}\left(\frac{R_\mathrm{p}}{2h}\right)^6.$$

The scattering amplitude is determined by the contribution of the monopole and dipole sources. The farther a particle from the surface ($M \to 0$), this expression coincides with the double amplitude of scattering by a free particle: the presence of a rigid boundary leads to the presence of a scattered field only in the upper half-space. The intensity of the monopole source does not depend on the distance to the boundary only in the considered lowest order of parameter $k(R_\mathrm{p}, h) \ll 1$.

DISCUSSION

The results presented above refer to the simplest case of a solid particle located near an impermeable boundary. The necessary prerequisites exist for generalization to the general case of an elastic particle near the boundary of two elastic media. The low-frequency asymptotic behavior of the Green's function for two elastic half-spaces has been found [18]. The approximate methods developed in this study are suitable for describing the general problem but require much more cumbersome calculations.

An important generalization is analysis of Rayleigh scattering by particles with a displaced center of mass, in particular, by "Janus" particles [19–21]. The new physical effect in this case is the excitation of rotational degrees of freedom. When describing scattering by nonspherical particles, the use of numerical solution methods is crucial. Along with the traditional method of T-matrices [22], alternative approaches can be used: diagrammatic equations [23] or discrete sources [24].

Under conditions where dissipative processes are unimportant, solving the linear problem makes it possible to describe a number of nonlinear (quadratic) effects, in particular, the radiation pressure force on a particle. Finding this force reduces to calculating the integral over the particle surface of a bilinear combination of solutions to a linear problem.

CONCLUSIONS

An analytical description of Rayleigh scattering is given for a rigid particle near an impenetrable boundary. The potential near a particle is described by the sum of the multipoles centered on the particle and its mirror image. The intensity of the multipoles is determined by the ratio of the particle radius to the distance to the interface surface. The scattering amplitude, which characterizes the far field, has the same dependence on the wavenumber as for scattering by a free particle and consists of the contributions from the monopole and dipole components. Only the dipole component depends on the location of the particle. The vibrational velocity of the center of the particle depends on the distance to the boundary by means of the effective inertial mass of the fluid.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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