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Letter

Wave fields under the influence of a random-driven force: The Burgers equation

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ABSTRACT

In this work, we examine the classical Burgers equation and investigate the effects of a random force on the wave field. Two scenarios are considered: the impact of a random force on different wave fields within the viscous Burgers equation and the effect of a periodic random force in the inviscid Burgers equation. For the first case, we demonstrate that the random force primarily causes wave fronts to increase or decrease depending on the dispersion parameter. For an initially deformed sinusoidal wave, the external force causes the mean wave field to spread out and dampen over time. The Cole-Hopf transformation is also used to obtain asymptotically the averaged wave field in certain regimes. For the inviscid problem, we assume the random force to be periodic with random phase to show that the mean wave field corresponds to the solution of the classical inviscid Burgers equation without external forces.

1. Introduction

The viscous Burguers equation in canonical form

$$\eta_t + \eta\eta_x - \nu\eta_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in (0,\infty), \tag{1}$$

with intial condition

$$\eta(x,0) = \eta_0(x), \quad x \in \mathbb{R}, \tag{2}$$

is a well-known model that appears in various branches of nonlinear science, including hydrodynamics, nonlinear acoustics, traffic flow, gas dynamics, and turbulence [4,7,24,49]. Here, $\eta(x,t)$ represents the surface wave elevation at position x and time t. The coefficient $\nu > 0$ measures the dissipation, which prevents wave breaking [49]. In the absence of viscosity ($\nu = 0$), it represents the dispersionless limit of significant equations in physics, such as the Korteweg-de Vries (KdV) equation, the Benjamin-Ono (BO) equation, the Whitham equation, and others [5,51,3,49,14,6,35]. Additionally, the model serves as a prototype for studying shock waves [4] which also appear in the study of plasma physics [1,34,36–39,45] and hydrodynamics [41,42,44]. Although the Burgers equation has been extensively studied in the literature, it continues to be a subject of ongoing research in the field of nonlinear physics [32].

An important tool commonly used to solve equation (1) is the Cole-Hopf transformation. This transformation converts problem (1) into the classical linear heat equation through the change of variables

$$\eta = -2\nu\frac{\varphi_x}{\varphi}, \tag{3}$$

where φ satisfies the heat equation

$$\varphi_t = \nu\varphi_{xx}. \tag{4}$$

Therefore, the solution to this equation can be mapped to the solution of equation (1). While it is not always possible to obtain closed-form expressions for the solutions for all choices of $\varphi(x,0)$, there are cases where it is possible. For instance, through the Cole-Hopf transformation, it is possible to show that the traveling wave fronts in the form [4,24,49]

$$\eta(x,t) = c - \frac{\eta_0^+ - \eta_0^-}{2}\tanh\left[\frac{\eta_0^+ - \eta_0^-}{4\nu}(x-ct)\right], \tag{5}$$

are solutions to equation (1). Here, the initial data η_0 satisfies $\eta_0(-\infty) = \eta_0^+$, $\eta_0(+\infty) = \eta_0^-$, and $\eta_0' < 0$, and the speed $c = (\eta_0^+ - \eta_0^-)/2$. Blowing

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up solutions in finite time can also be obtained through the Cole-Hopf transformation by taking

$$\varphi(x,t) = \exp(-\nu k^2 t)\cos(kx). \tag{6}$$

Consequently, the corresponding solution to equation (1) is

$$u(x,t) = 2\nu k \tan(kx). \tag{7}$$

Another interesting case involves negative viscosity, which leads to an increase in wave motion energy. This phenomenon has been extensively studied across various fields, including electrodynamics, plasma physics, atmospheric science, ocean circulation theory, and open channel flow dynamics [26,28,30,33,43]. A particular solution is given by

$$\eta(x,t) = \frac{U_0 \exp(-\nu k^2 t)\sin(kx)}{1 + \frac{R}{2}\exp(-\nu k^2 t)\cos(kx)}, \tag{8}$$

where $R = U_0/(k\nu) < 2$ [30]. For a more comprehensive discussion of solutions to equation (1), see the book by Gurbatov et al. [24].

The aim of this paper is to study the influence of a random force in the wave field. This is accomplished by considering the stochastic equation

$$\begin{aligned} &\eta_t + \eta\eta_x - \nu\eta_{xx} = f(x,t), \quad x \in \mathbb{R}, \quad t \in (0,\infty), \\ &\eta(x,0) = \eta_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{9}$$

where $f(x,t)$ is a stochastic process. External forces typically account for the influence the wave dynamics, for instance they are related to variable atmospheric pressure and topographic obstacles [9,12–14,16,17,21–23]. However, there are cases in which such that the external force is not deterministic [8,25]. The Burgers equation in the presence of a random force has appeared in the literature in different contexts. For instance, it has been used to investigate the Kolmogorov-Burgers turbulence [7,11,50]. Steady solutions to the forced Burgers equation have also been obtained when the driving force is a sinusoid [29] and theoretical results for a specific type of external force [40]. For a pure time-dependent random process, $f(x,t) = f(t)$, the problem has been discussed in many nonlinear dispersive equations [27,31,46–48,52]. However, to the best of our knowledge this is the first time that such a study is conducted within the Burgers equation framework. The main mathematical tool used in these papers to address randomness is to exploit the quadratic nonlinearity of the equation. By making a change of variables, the randomness is effectively transferred to the wave phase.

In this work, we compute the mean wave field under the influence of a random force, modeled as $f(x,t) = f(t)$. We consider examples where analytical solutions are available in the literature, utilizing the Cole-Hopf transformation. Two scenarios are considered: the impact of a random force on wave fronts and a deformed sinusoidal wave in the viscous Burgers equation and the effect of a periodic random force in the inviscid Burgers equation. For the first case, we demonstrate that the random force primarily causes an increase or decrease in the wave front depending on the dispersion parameter. Regarding the deformed sinusoidal wave, we demonstrate that the wave field tends to spread out and dampen over time. For the inviscid problem, we assume the random force to be periodic with random phase to show that the mean wave field corresponds to the solution of the classical inviscid Burgers equation without external forces

For reference, the article is organized as follows: In Section 2, the stochastic Burgers equation is reduced to the Burgers equation with constant coefficients. Results are presented in Sections 3-6, and the final conclusions are in Section 7.

2. Reducing the stochastic Burgers into a deterministic problem

Many authors have taken advantage of the fact that, with an appropriate change of variables, dispersive equations with quadratic nonlinearity of the form uu_x and a stochastic force can be transformed into equations with constant coefficients [18,25,52]. Consider the new variable

$$\eta(x,t) = u(x,t) + Z(t), \text{ where } Z'(t) = f(t). \tag{10}$$

Substituting equation (10) into equation (9) we obtain

$$u_t + Z(t)u_x + uu_x - \nu u_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in (0,\infty). \tag{11}$$

Now, we introduce the traveling variable

$$x' = x - V(t), \text{ where } V'(t) = Z(t). \tag{12}$$

Inserting equation (12) into equation (11), dropping the prime yields the homogenous viscous Burgers equation with constant coefficients

$$u_t + uu_x - \nu u_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in (0,\infty). \tag{13}$$

In the following sections, we consider the Cauchy problem with an initial wave front and other wave fields.

3. Wave fronts

Wave fronts play a crucial role in the study of undular bores and dispersive shock wave (DSW) theory. DSWs, also referred to as undular bores in fluid mechanics, represent a broad class of wave phenomena that emerge as solutions to nonlinear dispersive wave equations. They form as a result of the dispersive regularization of wave-breaking singularities or initial jump discontinuities, contrasting with the viscous shocks in compressible flow, such as those described by the viscous Burgers equation, where viscosity smooths the shock. Typically, DSWs manifest as modulated wavetrains, with solitary waves at one end and linear dispersive waves at the other, connecting two distinct flow states. This structure highlights a range of nonlinear behaviors within a coherent wave pattern. Notably, in the dispersionless limit of several dispersive equations, the Burgers equation serves as a prototype model [41,42,44].

The previously introduced change of variables shows that the random force primarily influences the wave phase and its pedestal. In other words, if

$$u(x,t) = c - \frac{u_0^+ - u_0^-}{2}\tanh\left[\frac{u_0^+ - u_0^-}{4\nu}(x - ct)\right] \tag{14}$$

is the solution of (13) then

$$\eta(x,t) = Z(t) + c - \frac{u_0^+ - u_0^-}{2}\tanh\left[\frac{u_0^+ - u_0^-}{4\nu}(x - V(t) - ct)\right] \tag{15}$$

is the exact solution of problem (9).

In order to obtain results on the averaged wave field, we assume that $W(V,t)$ is the distribution density function of the random quantity V, which is assumed to be uniform given by

$$W(V,t) = \frac{1}{2\sigma_V}\begin{cases} 1 & |V| < \sigma_V, \\ 0 & |V| > \sigma_V, \end{cases} \tag{16}$$

where $\sigma_V(t) = \sigma_0 t^\gamma$, with σ_0 and γ constants. With this choice, the wave behaves as a Brownian particle. Notice that the pedestal $Z(t)$ does not affect the intrinsic properties of the wave front. The randomness primarily affects the wave phase. Therefore, the averaged wave field is obtained by averaging the wave described in equation (14). Consequently, the mean field is

$$< u(x,t) >= \frac{1}{2\sigma_V}\int_{-\sigma_V}^{\sigma_V} u(x,t)dV. \tag{17}$$

Substituting the expression (14) into equation (17) yields

$$< u(x,t) >= c - \frac{\nu}{2}\log\left[\cosh\left(\frac{u_0^+ - u_0^-}{2}(x - ct - V)\right)\right]\Big|_{-\sigma_V}^{\sigma_V}. \tag{18}$$

Therefore,

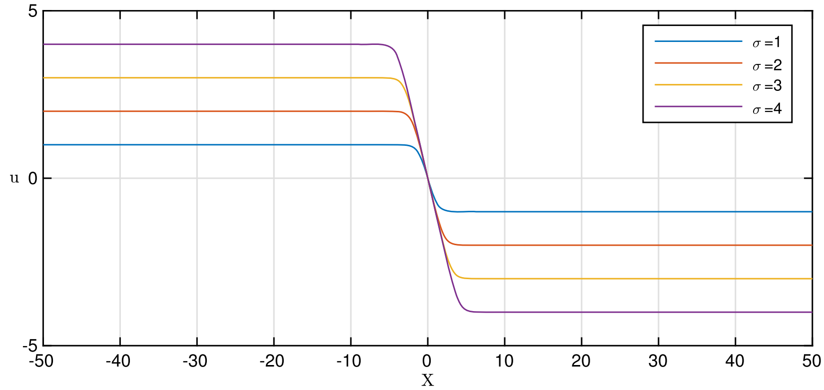

Fig. 1. The averaged wave front given by equation (23) for different values of σ.

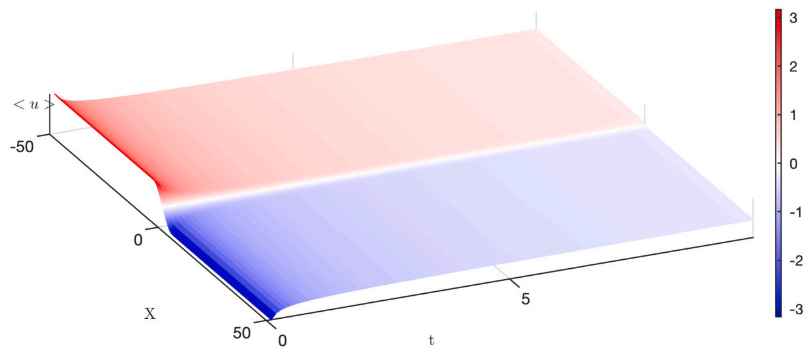

Fig. 2. The mean averaged wave front over time (23). Parameters: $\sigma_0 = 1$ and $\gamma = -0.5$.

$$< u(x,t) >= c - \frac{\nu}{2}\log\left[\frac{\cosh\left(\frac{u_0^+ - u_0^-}{2}(x - ct + \sigma_V)\right)}{\cosh\left(\frac{u_0^+ - u_0^-}{2}(x - ct - \sigma_V)\right)}\right]. \tag{19}$$

Introducing the variables

$$X = \frac{u_0^+ - u_0^-}{2}(x - ct), \quad \sigma = \frac{u_0^+ - u_0^-}{2}\sigma_V, \tag{20}$$

the mean wave front can be written as

$$< u(X,t) >= c - \frac{\nu}{2}\log\left[\frac{\cosh(X + \sigma)}{\cosh(X - \sigma)}\right]. \tag{21}$$

Rescaling the averaged wave front as

$$< u(X,t) > \to \frac{< u(X,t) > - c}{\nu}, \tag{22}$$

allow us to write the averaged wave field as a function of X and σ

$$< u(X,\sigma) >= -\frac{1}{2}\log\left[\frac{\cosh(X + \sigma)}{\cosh(X - \sigma)}\right], \tag{23}$$

and consequently analyses the effect of the dispersion parameter on the mean wave field. Notice yet that L'Hospital rule implies

$$\lim_{X\to\infty} < u(X,\sigma) >= 2\sigma. \tag{24}$$

Fig. 1 shows the averaged wave field for different values of the dispersion (σ) parameter. As we can see as the dispersion parameter increases the wave front becomes higher. However, the steepness of the wave front remains the same for different values of σ.

Now, we examine the evolution of the mean wave front over time. When γ is negative, the amplitude of the wave front decays over time, resulting in a progressively weaker wave front (Fig. 2). Conversely, when γ is positive, the amplitude of the wave front increases over time and can grow indefinitely. This behavior is illustrated in Fig. 3. From a physical standpoint, in this case, the wave front can potentially cause significant damage to the region it propagates through, as seen with flooding waves.

4. Deformed sinusoidal wave

The evolution of a single harmonic is ideal for investigating the primary effects of nonlinearity in a wave field. This approach was first explored by Zabusky and Kruskal [51] and has been applied in other contexts, such as the recurrence problem initially studied by Fermi et al. [10], where a chain of particles with equal masses is connected by elastic springs governed by a dynamical system with quadratic and cubic spring forces [2,10]. This study has also been used in other frameworks [19,20] and more recently to describe the impact of a horizontal electric field on rotational flows [15]. With this in mind, we investigate the impact of randomness on the solutions represented by the deformed sinusoidal wave described in the equation (8). According to the change of variables defined in Section 2 we have

$$u(x,t) = \frac{U_0\exp(-\nu k^2 t)\sin(k(x - V(t)))}{1 + \frac{R}{2}\exp(-\nu k^2 t)\cos(k(x - V(t)))}, \tag{25}$$

where V has the probability density defined in (16). Fig. 4 represents the solution (25) for specific choices of the parameters. As we can see, at time $t = 0$, we have a slightly deformed sinusoidal wave periodic in space, and as time progresses its amplitude is damped over time. The averaged wave field is obtained by integrating equation (25) in the interval $[-\sigma_V, \sigma_V]$. Consequently, we have the following expression for the mean wave field

$$< u(x,t) >= \frac{2U_0}{Rk}\log\left[\frac{\frac{R}{2}\exp(-\nu k^2 t)\cos(k(x - \sigma_V)) + 1}{\frac{R}{2}\exp(-\nu k^2 t)\cos(k(x + \sigma_V)) + 1}\right]. \tag{26}$$

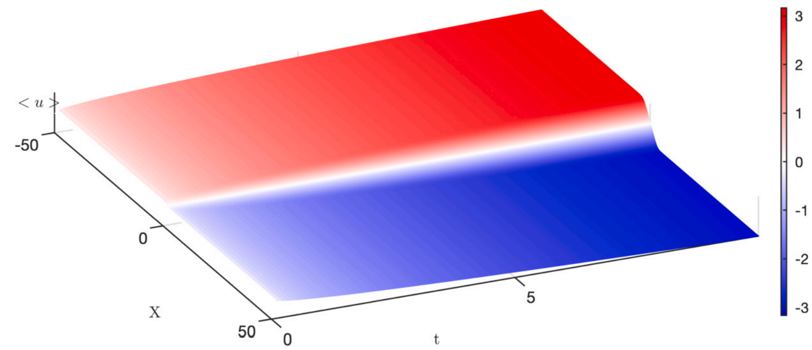

Fig. 3. The mean averaged wave front over time (23). Parameters: $\sigma(t) = \sigma_0 t^\gamma$ with $\sigma_0 = 1$ and $\gamma = 0.5$.

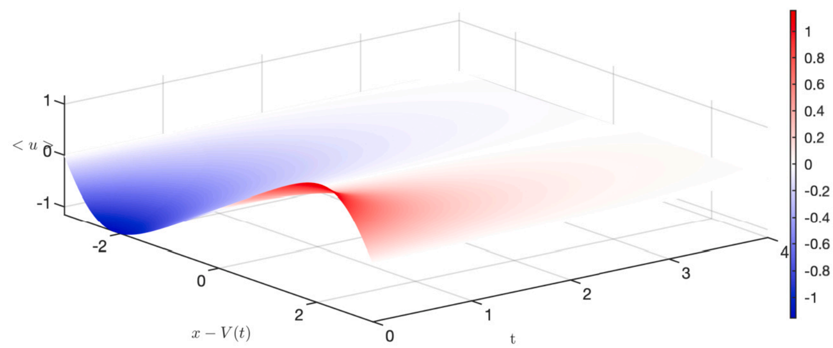

Fig. 4. Typical deterministic solution given by equation (25) in the moving frame. Parameters: $k = 1$, $U_0 = 1$ and $\nu = -1$.

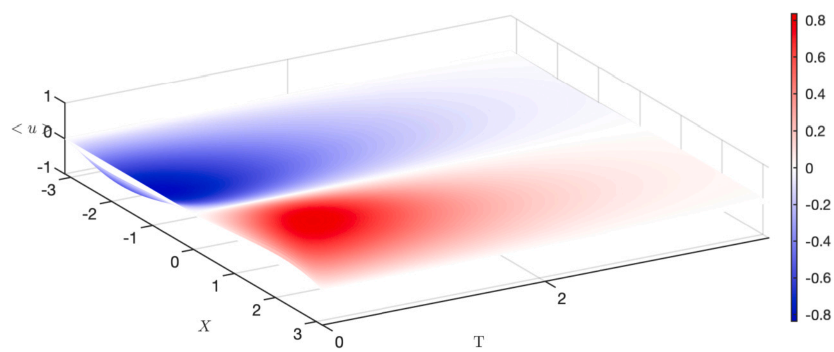

Fig. 5. The averaged solution given by equation (27). Parameters: $R = 1$, $\sigma(t) = \sigma_0 t^\gamma$ with $\sigma_0 = 1$ and $\gamma = 0.5$.

Fig. 5 represents the averaged wave field for a choice of parameters. Rescaling the variables as $X = kx$, $\sigma = k\sigma_V$, $T = \nu k^2 t$ and $< u(x,t > /(2U_0/Rk) \to < u(X,T) >$ yields

$$< u(X,T) >= \log\left[\frac{\frac{R}{2}\exp(-T)\cos(X-\sigma)+1}{\frac{R}{2}\exp(-T)\cos(X+\sigma)+1}\right] \tag{27}$$

The parameter γ is fundamental in describing the mean wave field, and its influence can be divided into two cases: $\gamma < 1$ and $\gamma > 1$. For $\gamma < 1$, the dispersion causes the deformed sinusoidal wave to spread out along the x-axis while also damping over time, as illustrated in Fig. 5. This behavior is akin to the spreading and damping observed in solitons for the KdV equation [31] and the Benjamin-Ono equation [18]. In contrast, for $\gamma > 1$, we observe a more oscillatory behavior in the mean field, resulting in a "chess pattern" appearance, as shown in Fig. 6.

5. Cole-Hopf transformation in random fields

We recall the classical Cole-Hopf transformation

$$u = -2\nu\frac{\varphi_{x'}}{\varphi}, \tag{28}$$

where φ satisfies the heat equation

$$\varphi_t = \nu\varphi_{x'x'}. \tag{29}$$

The general solution of the heat equation is written in form

$$\varphi(x',t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{\mathbb{R}}\mathbf{\Phi}(\eta)\exp\Big(-\frac{(x'-\eta)^2}{4\nu t}\Big)d\eta, \tag{30}$$

with initial condition

$$\varphi(x',0) = \mathbf{\Phi}(x') = \exp\Big[-\frac{1}{2\nu}\int_0^{x'}u(\eta,0)d\eta\Big]. \tag{31}$$

Such solution is mapped into the following solution to the viscous Burgers equation

$$u(x',t) = \frac{\int_{\mathbb{R}}\frac{x'-\eta}{t}e^{-G/2\nu}d\eta}{\int_{\mathbb{R}}e^{-G/2\nu}d\eta} \tag{32}$$

where,

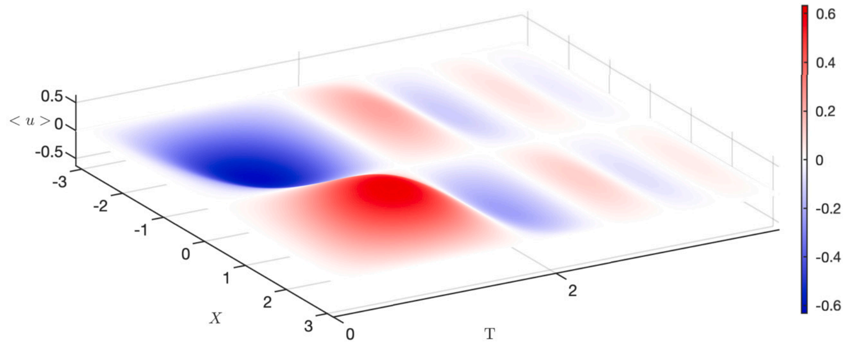

Fig. 6. The averaged solution given by equation (27). Parameters: $R = 1$, $\sigma(t) = \sigma_0 t^\gamma$ with $\sigma_0 = 1$ and $\gamma = 2$.

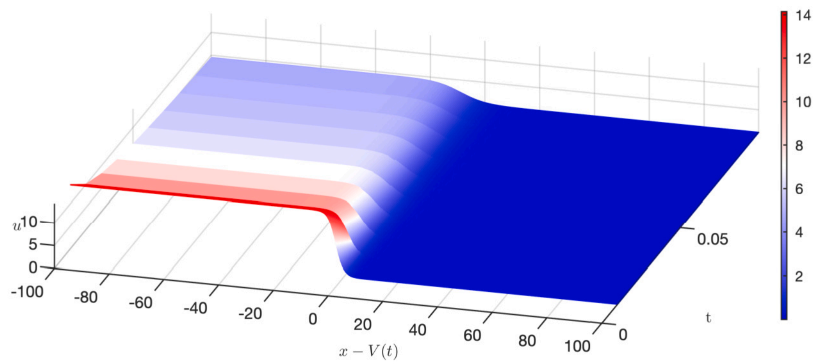

Fig. 7. The solution given in equation (41) in the moving frame with speed $V(t)$. Parameters: $A = 1$ and $\nu = 0.1$.

$$G(\eta, x', t) = \int_0^\eta u(\eta', 0) d\eta' + \frac{(x' - \eta)^2}{2t}. \tag{33}$$

For more details we recommend the reader to see [49]. In general, we cannot write the exact solution in closed form. However, there are special cases in which we can use asymptotic results to obtain a solution. Assuming that the initial data has the form

$$u(x', 0) = A\delta(x') \tag{34}$$

and that the ratio $R = A/2\nu$ is small, the following approximation is valid [49]

$$u(x', t) = \frac{A}{\sqrt{4\pi\nu t}} e^{-\frac{x'^2}{4\nu t}}. \tag{35}$$

Consequently, we have

$$u(x, t) = \frac{A}{\sqrt{4\pi\nu t}} e^{-\frac{(x-V(t))^2}{4\nu t}}. \tag{36}$$

Assuming that the external force is a stationary random process with a zero mean and the Gaussian distribution function, we also write the distribution function $W(V, t)$ in the form of the Gaussian dependence as

$$W(V, t) = \frac{1}{\sqrt{2\pi}\sigma_V} \exp\Big(- \frac{V^2}{2\sigma_V^2} \Big). \tag{37}$$

The mean wave field becomes

$$< u(x, t) > = \int_{\mathbb{R}} \frac{A}{\sqrt{4\pi\nu t}} \exp\Big(- \frac{(x - V)^2}{4\nu t} \Big) \frac{1}{\sqrt{2\pi}\sigma_V} \exp\Big(- \frac{V^2}{2\sigma_V^2} \Big) dV. \tag{38}$$

Notice that if $\gamma < 1/2$, then the wave $u(x, t)$ spreads out faster the Gaussian (37), thus the second term that appears in the integral described in equation (38) can be considered as a delta function, consequently, the wave field can be approximate as

$$< u(x, t) > = \frac{A}{\sqrt{8\pi^2 \nu t \sigma_V}} \exp\Big(- \frac{x^2}{4\nu t} \Big). \tag{39}$$

This shows that the mean wave field is damped over time. On the other hand if $\gamma > 1/2$ the normal distribution spreads out faster and thus the wave can be approximated as a delta function concentrated at x. This yields the approximation

$$< u(x, t) > = \frac{A}{\sqrt{8\pi^2 \nu t \sigma_V}} \exp\Big(- \frac{x^2}{2\sigma_V^2} \Big). \tag{40}$$

Equations (39) and (40) demonstrate that diffusion to dominates over the nonlinearity.

Now we consider the case that $R = A/2\nu$ is large. In this case, we have the following approximation

$$u(x, t) = \sqrt{\frac{2A}{t}} \frac{1}{1 + \exp\Big[\frac{1}{2\nu} \sqrt{\frac{2A}{t}} (x - V - \sqrt{2At}) + \frac{1}{2} \log(\frac{2\pi A}{\nu}) \Big]}. \tag{41}$$

Fig. 7 displays a typical wave front for different parameter settings. The wave front typically spreads out due to diffusion mechanisms. For this case, the averaged wave field for $V \sim \mathcal{U}(-\sigma_V, \sigma_V)$ is obtained by integrating equation (41). Thus,

$$\begin{aligned} < u(x, t) > = 2\nu \Bigg[\frac{\sigma_V}{2\nu} \sqrt{\frac{2A}{t}} + \log\Bigg(e^{\frac{\sigma_V}{2\nu}\sqrt{\frac{2A}{t}}} + e^{\frac{1}{2\nu}\sqrt{\frac{2A}{t}}(x - \sqrt{2At}) + \frac{1}{2}\log(\frac{2\pi A}{\nu})} \Bigg) \\ - \log\Bigg(e^{\frac{\sigma_V}{2\nu}\sqrt{\frac{2A}{t}} + \frac{1}{2\nu}\sqrt{\frac{2A}{t}}(x - \sqrt{2At}) + \frac{1}{2}\log(\frac{2\pi A}{\nu})} + 1 \Bigg) \Bigg]. \end{aligned} \tag{42}$$

The mean field is qualitatively different from the example give above, see Fig. 8.

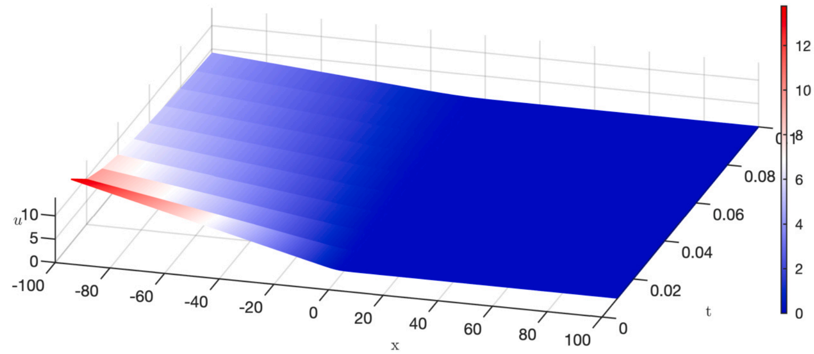

Fig. 8. The mean wave front given in equation (42). Parameters: $A=1$, $\nu=0.1$, $\sigma_0=1$, $\gamma=0.5$.

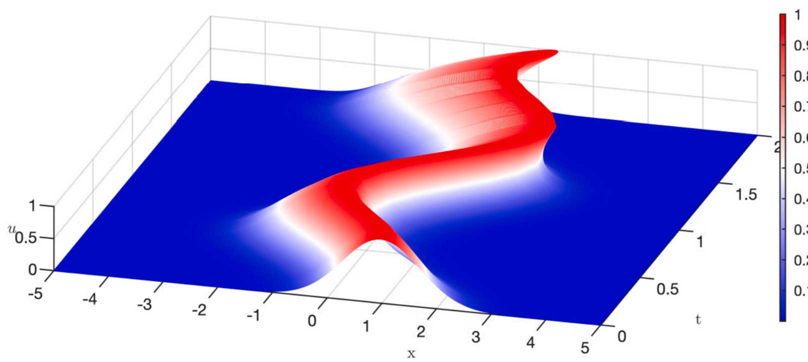

Fig. 9. The solution (47) of the Cauchy problem (44). Parameters: $A=1$, $\omega=5$.

6. Gaussian pulses on periodic fields

In this section we consider the inviscid Burgers equation with a random force. The function V is taken as

$$V(t)=A\cos(\omega t+\varphi), \tag{43}$$

where A and ω are constants and φ is the random phase, which is assumed to be uniformly distributed in the interval $[0,2\pi]$.

The method of characteristics can be used to obtain parametric solutions of the following Cauchy problem

$$\begin{aligned} &u_t+uu_{x'}=0, \quad x'\in\mathbb{R}, \quad t\in(0,\infty), \\ &u(s,0)=h(s), \quad s\in\mathbb{R}. \end{aligned} \tag{44}$$

Here, we assume that

$$h(s)=\exp(-s^2). \tag{45}$$

The exact solution to this problem can be parametrized as

$$x'(s,t)=s+h(s)t. \tag{46}$$

Using equation (12), we obtain the parametrized solution

$$x(s,t)=s+V(t)+h(s)t. \tag{47}$$

A typical example of the randomness introduce in the solution is given in Fig. 9 for specific values of A and ω. An oscillatory pattern in the wave crest is observed due to the periodicity of the phase, followed by the overturning of the wave field.

The mean field is obtained by averaging equation (47) for $\varphi\in[0,2\pi]$ as

$$<x(s,t)>=s+<V(t)>+h(s)t=s+h(s)t. \tag{48}$$

Therefore, the mean wave field is the solution of the homogeneous inviscid Burgers equation. At this scale, the external force does not introduce inhomogeneities into the wave field. It is well known that solutions to the inviscid Burgers equation can exhibit breaking, where the wave overturns, leading to a gradient catastrophe. However, in this context, we show that the periodic external force does not affect the mean wave field. The resulting averaged solution is depicted in Fig. 10. This shows that despite the presence of external random forces, the large-scale behavior of the wave field remains governed by the same dynamics as the inviscid Burgers equation, preserving its characteristic properties without additional complexities.

7. Conclusions and discussion

In this work, we investigated the effects of a random time-dependent force on wave fronts and Gaussian pulses within the framework of the viscous and inviscid Burgers equations, respectively. Analytically, we obtained the mean wave fields. For the viscous Burgers equation, we demonstrated that the random force can either increase or decrease the amplitude of the wave front depending on whether the dispersion parameter (σ) is increasing or decreasing. Meanwhile, for a deformed sinusoidal wave, the mean wave field spreads out and dampens over time. Additionally, we showed that the averaged wave field for Gaussian pulses in periodic random fields corresponds to the exact solution of the inviscid Burgers equation.

The primary limitation of this method is its reliance on the quadratic nonlinearity of the problem and the restriction of the external force to being time-dependent. When the external force depends on spatial variables, the expression for the exact solution to the Burgers equation becomes significantly more complex, making it difficult to compute the mean wave field analytically. However, this opens the door for numerical studies, which present a promising avenue for future research.

CRediT authorship contribution statement

Marcelo V. Flamarion: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptual-

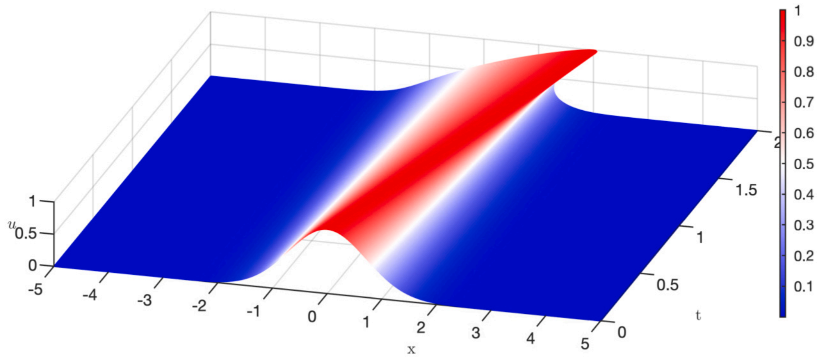

Fig. 10. The averaged Gaussian field.

ization. **Efim Pelinovsky:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization. **Denis V. Makarov:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data sharing is not applicable to this article as all parameters used in the numerical experiments are informed in this paper.

References

[1] W. Alhejaili, S. Roy, S. Raut, A. Roy, A.H. Salas, T. Aboelenen, S.A. El-Tantawy, Analytical solutions to (modified) Korteweg–de Vries–Zakharov–Kuznetsov equation and modeling ion-acoustic solitary, periodic, and breather waves in auroral magnetoplasmas, Phys. Plasmas 31 (2024) 082107.
[2] G.P. Berman, F.M. Izrailev, The Fermi-Pasta-Ulam problem: fifty years of progress, Chaos 15 (2004) 015104.
[3] T.B. Benjamin, Internal waves of permanent form of great depth, J. Fluid Mech. 295 (1967) 381–394.
[4] J.M. Burgers, The Nonlinear Diffusion Equation–Asymptotic Solutions and Statistical Problems, Springer, 1974.
[5] N. Cao, X.J. Yin, S. Bai, L.Y. Xu, Lump-soliton, rogue-soliton interaction solutions of an evolution model for magnetized Rossby waves, Nonlinear Dyn. 112 (2024) 9367–9389.
[6] N.M. Chadha, S. Tomar, S. Raut, Parametric analysis of dust ion acoustic waves in superthermal plasmas through non-autonomous KdV framework, Commun. Nonlinear Sci. Numer. Simul. 123 (2023) 107269.
[7] A. Checkhlov, V. Yakhot, Kolmogorov turbulence in a random-force-driven Burgers equation, Phys. Rev. E 51 (4) (1995) 2739–2742.
[8] A. Debussche, J. Printems, Numerical simulation of the stochastic Korteweg-de Vries equation, Physica D 134 (1999) 200–226.
[9] A. Ermakov, Y. Stepanyants, Soliton interaction with external forcing within the Korteweg-de Vries equation, Chaos 29 (2019) 013117.
[10] E. Fermi, J. Pasta, S. Ulam, Studies of the nonlinear problems, Phys. Rep. (1955).
[11] E. Weinan, K. Khanin, A. Mazel, Y. Sinai, Probability distribution functions for the random forced Burgers equation, Phys. Rev. Lett. 10 (1997) 1904–1907.
[12] M.V. Flamarion, Generation of trapped depression solitary waves in gravity-capillary flows over an obstacle, Comput. Appl. Math. 41 (2022) 31.
[13] M.V. Flamarion, E. Pelinovsky, Soliton interactions with an external forcing: the modified Korteweg-de Vries framework, Chaos Solitons Fractals 165 (2022) 112889.
[14] M.V. Flamarion, Trapped waves generated by an accelerated moving disturbance for the Whitham equation, Partial Differ. Equ. Appl. Math. 5 (2022) 100356.
[15] M.V. Flamarion, E. Kochurin, R. Ribeiro-Jr, N. Zubarev, Fully nonlinear evolution of free-surface waves with constant vorticity under horizontal electric fields, Mathematics 11 (2023) 4467.
[16] M.V. Flamarion, E. Pelinovsky, Solitary wave interactions with an external periodic force: the extended Korteweg-de Vries framework, Mathematics 10 (2022) 4538.
[17] M.V. Flamarion, E. Pelinovsky, Interaction of interfacial waves with an external force: the Benjamin-Ono equation framework, Symmetry 15 (2023) 1478.
[18] M.V. Flamarion, E. Pelinovsky, Solitons in dissipative systems subjected to random force within the Benjamin-Ono type equation, Chaos Solitons Fractals 187 (2024) 115373.
[19] K. Goda, Numerical studies on recurrence of the Korteweg-de Vries equation, J. Phys. Soc. Jpn. 42 (3) (1977) 1040–1046.
[20] K. Goda, Numerical studies of the regularized long wave equation, J. Phys. Soc. Jpn. 48 (2) (1980) 623–630.
[21] R. Grimshaw, B.A. Malomed, X. Tian, Dynamics of a KdV soliton due to periodic forcing, Phys. Lett. A 179 (1993) 291–298.
[22] R. Grimshaw, E. Pelinovsky, X. Tian, Interaction of a solitary wave with an external force, Physica D 77 (1994) 405–433.
[23] R. Grimshaw, E. Pelinovsky, P. Sakovl, Interaction of a solitary wave with an external force moving with variable speed, Stud. Appl. Math. 142 (1996) 433–464.
[24] S.N. Gurbatov, O.V. Rudenko, A.I. Saichev, Waves and Structures in Nonlinear Nondispersive Media–General Theory and Applications to Nonlinear Acoustics, Springer, 2011.
[25] S.N. Gurbatov, E.N. Pelinovskii, A.I. Saichev, Problem of closure of the equations for the averaged fields in nonlinear media containing chaotic inhomogeneities, Radiophys. Quantum Electron. 21 (1978) 1032–1037.
[26] R.S. Johnson, Shallow water waves on a viscous fluid –the undular bore, Phys. Fluids 15 (2) (1972) 1693–1699.
[27] Q. Liu, Some exact solutions for stochastic mKdV equation, Chaos Solitons Fractals 32 (3) (2007) 1224–1230.
[28] S.A. Mashkovich, I.G. Veil', Large scale atmospheric processes with a "negative viscosity", Meteorol. Gidrol. (1970) 8.
[29] M. Okamura, T. Kawahara, Steady solutions of forced Burgers equation, J. Phys. Soc. Jpn. 11 (1983) 3800–3806.
[30] E.N. Pelinovskii, V.E. Fridman, Explosive instability in nonlinear waves in media with negative viscosity, PMM 38 (6) (1974) 991–995.
[31] E. Pelinovsky, A. Sergeeva, KdV-soliton dynamics in a random field, Radiophys. Quantum Electron. 49 (7) (2006) 540–546.
[32] E. Pelinovsky, T. Talipova, E. Didenkulova, The Hopf equation with certain modular nonlinearities, Phys. Lett. A 507 (2024) 129489.
[33] M.I. Rabinovich, V.P. Reutov, Interaction of parametrically coupled waves in nonequilibrium media (review), Radiophys. Quantum Electron. 16 (6) (1973) 625–643.
[34] S. Raut, S. Roy, R.R. Kairi, P. Chatterjee, Approximate analytical solutions of generalized Zakharov–Kuznetsov and generalized modified Zakharov–Kuznetsov equations, Int. J. Appl. Comput. Math. 7 (2021) 157.
[35] S. Raut, S. Roy, S. Saha, A.N. Das, Studies on the dust–ion–acoustic solitary wave in planar and non-planar super-thermal plasmas with trapped electrons, Plasma Phys. Rep. 48 (2022) 627–637.
[36] S. Raut, K.K. Mondal, P. Chatterjee, S. Roy, Dust ion acoustic bi-soliton, soliton, and shock waves in unmagnetized plasma with Kaniadakis-distributed electrons in planar and nonplanar geometry, Eur. Phys. J. D 77 (2023) 100.
[37] A. Roy, K.K. Mondal, P. Chatterjee, S. Raut, Influence of external periodic force on ion acoustic waves in a magnetized dusty plasma through forced KP equation and modified forced KP equation, Braz. J. Phys. 52 (2022) 65.
[38] H. Schamel, Stationary solitary, snoidal and sinusoidal ion acoustic waves, Phys. Plasmas 14 (1972) 905.
[39] H. Schamel, A modified Korteweg-de Vries equation for ion acoustic waves due to resonant electrons, J. Plasma Phys. 9 (3) (1973) 377–387.

[40] Y.G. Sinai, Two results concerning asymptotic behavior of solutions of the Burgers equation with force, J. Stat. Phys. 64 (1991) 1–12.
[41] A. Porter, N. Smyth, Modelling the morning glory of the Gulf of Carpentaria, J. Fluid Mech. 454 (2002) 1–20.
[42] N.F. Smyth, P.E. Holloway, Hydraulic jump and undular bore formation on a shelf break, J. Phys. Oceanogr. 18 (1988) 947–962.
[43] V. Starr, Physics of Negative Viscosity Phenomena, McGraw, New York, 1968.
[44] R.M. Vargas-Magaña, T.R. Marchant, N. Smyth, Numerical and analytical study of undular bores governed by the full water wave equations and bidirectional Whitham-Boussinesq equations, Phys. Fluids 33 (2021) 067105.
[45] M.K. Verma, S. Chatterjee, A. Sharma, A. Mohapatra, Equilibrium states of Burgers and Korteweg-de Vries equations, Phys. Rev. E 105 (2022) 034121.
[46] M. Wadati, Stochastic KdV equation, J. Phys. Soc. Jpn. 52 (1990) 2642–2648.
[47] M. Wadati, Y. Akutsy, Stochastic KdV equation with and without damping, J. Phys. Soc. Jpn. 301 (1984) 3342–3350.
[48] M. Wadati, Deformation of solitons in random media, J. Phys. Soc. Jpn. 59 (12) (1990) 4201–4203.
[49] G.B. Whitham, Linear and Nonlinear Waves, John Wiley & Sons, Inc, New York, 1974.
[50] V. Yakhot, Z.S. She, Long-time, large-scale properties of the random-force-driven Burgers equation, Phys. Rev. Lett. 60 (18) (1988) 1840.
[51] M. Zabusky, N. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965) 240–243.
[52] N. Zahibo, E. Pelinovsky, A. Sergeeva, Weakly damped KdV soliton dynamics with the random force, Chaos Solitons Fractals 39 (2009) 1645–1650.