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Spectral Problem for a Harmonic Chain with Dissipation at the Boundaries*

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Abstract—We consider a spectral problem for a dynamical system describing, in the Schrödinger variables, the motion in a finite homogeneous chain of coupled harmonic oscillators with boundary conditions that admit dissipation and energy pumping in the system. The solution of this problem is given for arbitrary parameter values in the boundary conditions and for any sufficiently large number of oscillators in the chain.

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1. INTRODUCTION

A chain of coupled harmonic oscillators, or, briefly, a harmonic chain, is the simplest physical body model serving to describe one-dimensional oscillatory processes in the harmonic approximation [1]. For example, this model is well known to be used as early as by Newton to calculate the speed of sound in air [2]. Its present popularity is related to attempts to substantiate Fourier's law of thermal conductivity [3], [4].

The adequate application of this model is hindered by the lack of known exact solutions in a number of cases relevant for physical applications, for example, when dissipation is admitted at the ends of the chain. This case is important, for example, when modelling heat baths on the boundaries of a solid [3], [5]. In the absence of exact solutions, the structure of the eigenoscillations of the chain and the description of invariant subspaces of its dynamic equations come to the fore. In other words, solving the spectral problem for the harmonic chain becomes a must. The solution of this problem permits one to make conclusions about the types of motion in the chain, say, from the viewpoint of stability theory and provides opportunities for obtaining solutions of the chain dynamic equations in acceptable form.

The present paper deals with the spectral problem for the dynamical system of a homogeneous harmonic chain with dissipation and/or energy pumping at the boundaries. Mathematically, this problem belongs to the class of matrix spectral problems with a tridiagonal matrix. For example, the classical matrix spectral problem deals with a Jacobi matrix, which is a symmetric tridiagonal matrix with positive off-diagonal entries. It is this type of matrix that is associated with the dynamical system of a (generally, inhomogeneous) harmonic chain. This class of problems has been widely studied, for example, in connection with problems of oscillation theory [6], [7], inverse spectral problems [7]–[11], the theory of orthogonal polynomials [12], [13], and a number of other problems.

However, just as in the case of the harmonic chain, matrix spectral problems have mainly been considered for the conservative case. The spectral properties of dissipative Jacobi matrices have turned out to be difficult to study. For example, the solution of the inverse spectral problem for a Jacobi matrix

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with dissipation has been obtained relatively recently and only for the case in which the dissipation is specified at one of the ends of the main diagonal of the matrix [14], [15]. Apparently, the solution of this problem for the case in which the dissipation is specified at both ends of the main diagonal is yet to be found, and in our opinion, this is due to the lack of an adequate solution of the direct spectral problem. Thus, it is of interest first to solve the spectral problem for the case of a homogeneous matrix with dissipation at two ends.

We solve this problem in the following sense. The location of the roots of the characteristic polynomial of the system matrix on the complex plane is described for arbitrary parameter values in the boundary condition and for any sufficiently large number of oscillators in the chain. Conditions for these roots to be simple are obtained, and the stability domain of the characteristic polynomial is found. We describe the eigenvectors of the system matrix and establish their orthogonality and completeness.

Our approach to the problem has the following specific feature. Already in the original dynamic equations, we replace the natural variables describing the oscillator displacements from the equilibrium position by the Schrödinger variables [16]–[18], which describe the relative displacements of neighboring oscillators and the oscillator velocities. After this replacement, the chain equations become a linear dynamical system whose matrix, in the absence of dissipation, is a tridiagonal antisymmetric matrix. The addition of dissipation leads to the occurrence of nonzero diagonal entries. In our opinion, the analysis of the spectral problem for such a matrix is more natural than for the original problem.

Apart from the introduction, the paper contains Sec. 2, where the statement of the problem is given, and four main sections: on the location of the roots, their multiplicity, the stability of the characteristic polynomial, and the orthogonality and completeness of the system of eigenvectors (Secs. 3–6). The appendix (Sec. 7) presents the system of equations of the harmonic chain in the original natural variables and the transformation of this system to the Schrödinger variables.

2. STATEMENT OF THE PROBLEM

The motion of oscillators in a homogeneous harmonic chain with dissipation and pumping at the boundaries is described in the Schrödinger variables by the linear dynamical system (see Sec. 7)

$$\dot{x}_l = x_{l+1} - x_{l-1}, \quad l = 0, \dots, N-1, \tag{2.1}$$

$$x_{-1} + bx_0 = 0, \quad x_N - cx_{N-1} = 0, \qquad b, c \in \mathbb{R}, \tag{2.2}$$

or, in matrix form, $\dot{x} = Ax$, where

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{pmatrix}, \quad A = \begin{pmatrix} b & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & c \end{pmatrix}.$$

The spectral problem associated with this dynamical system is given by the system of equations

$$\begin{aligned} \lambda y_l &= y_{l+1} - y_{l-1}, \quad l = 1, \dots, N-2, \\ \lambda y_0 &= by_0 + y_1, \\ \lambda y_{N-1} &= -y_{N-2} + cy_{N-1}, \end{aligned} \tag{2.3}$$

or, in matrix form,

$$A\gamma = \lambda\gamma, \qquad \gamma = \begin{pmatrix} y_0 \\ \dots \\ y_{N-1} \end{pmatrix}.$$

We represent (2.3) in a form similar to (2.1), (2.2), that is, as the boundary value problem

$$\lambda y_l = y_{l+1} - y_{l-1}, \quad l = 0, \dots, N-1, \tag{2.4}$$

$$y_{-1} + by_0 = 0, \quad y_N - cy_{N-1} = 0, \qquad b, c \in \mathbb{R}. \tag{2.5}$$

Remark 1. In operator form, following, e.g., [19], system (2.3) is represented as the spectral problem

$$\mathcal{A}y = \lambda y$$

for the second-order difference operator

$$\begin{aligned} \mathcal{A}y_l &= y_{l+1} - y_{l-1}, \qquad l = 1, \dots, N-2, \\ \mathcal{A}y_0 &= y_1 - y_{-1} + by_0, \\ \mathcal{A}y_{N-1} &= y_N - y_{N-2} + cy_{N-1} \end{aligned} \tag{2.6}$$

acting on the space of functions of the integer argument $l = -1, \dots, N$ with the Dirichlet boundary conditions

$$y_{-1} = y_N = 0.$$

In formula (2.6), the notation

$$\mathcal{A}y_l = (\mathcal{A}y)_l, \qquad y_l = y(l)$$

is used.

The aim of the present paper is to study the eigenvalues and eigenvectors of the matrix A for arbitrary values of the parameters b and c and any sufficiently large N depending on these parameters. In particular, we intend to describe the eigenvalue location on the complex plane as $N \to \infty$, determine the conditions for the eigenvalues to be simple, find the stability domain of the characteristic polynomial of the matrix A, and establish conditions for the orthogonality and completeness of the eigenvector system.

This problem can conveniently be solved using the technique of finite orthogonal polynomials [10], [12], [13]. Thus, we introduce the sequence of polynomials $p_l(\lambda)$, $l = -1, \dots, N$, whose elements are defined by the expressions

$$\begin{aligned} p_l &= y_l, \qquad l = -1, \dots, N-1, \\ p_N &= y_N - cy_{N-1}, \end{aligned} \tag{2.7}$$

where y_l, $l = -1, \dots, N$, is the solution of the recurrence relation

$$\lambda y_l = y_{l+1} - y_{l-1}, \qquad l = 0, 1, \dots, \tag{2.8}$$

with the initial conditions

$$y_{-1} = -b, \qquad y_0 = 1.$$

In terms of these polynomials, the eigenvalues and eigenvectors are described as follows.

Proposition 1. *One has*

$$p_l(\lambda) = (-1)^l \det(A_l - \lambda I), \quad l = 1, \dots, N, \tag{2.9}$$

where A_l is the lth principal leading submatrix of A and I is the identity matrix of appropriate size. Therefore, $p_N(\lambda)$ is the characteristic polynomial of A, and the eigenvalues of A are the roots of this polynomial.

The eigenvectors of the matrix A can be represented in the form

$$\gamma_k = \begin{pmatrix} p_0(\lambda_k) \\ \dots \\ p_{N-1}(\lambda_k) \end{pmatrix}, \qquad 0 \le k \le N-1, \tag{2.10}$$

where the λ_k are the eigenvalues of A.

Proof. Expanding the determinant in (2.9) along the last row, we see that for $l < N$ both sides of this relation are solutions of the same initial value problems for Eq. (2.8). This proves (2.9) for $l \neq N$ in view of the uniqueness of the solution. Now Eq. (2.9) for $l = N$ follows from (2.7). The expression (2.10) follows from (2.7) and (2.8). □

Along with the polynomials $p_l(\lambda)$, we use the polynomials $P_l(\lambda)$ defined as $p_l(\lambda)$ with $b = c = 0$. Then one has

$$\begin{aligned} p_l &= P_l - bP_{l-1}, \\ p_N &= P_N - (b+c)P_{N-1} + bcP_{N-2}, \qquad l = 0, \dots, N-1. \end{aligned} \tag{2.11}$$

It is convenient to represent the polynomials $p_l(\lambda)$ as functions of the spectral variable z related to the natural spectral variable λ by the formula

$$\lambda = z - \frac{1}{z}.$$

Lemma 1. *One has*

$$P_l\left(z - \frac{1}{z}\right) = \frac{z^{2l+2} + (-1)^l}{z^l(1+z^2)}, \quad l = -1, \dots, N-1, \tag{2.12}$$

and consequently,

$$p_N\left(z - \frac{1}{z}\right) = \frac{z^{2N}(z-b)(z-c) + (-1)^N(1+bz)(1+cz)}{z^N(1+z^2)}. \tag{2.13}$$

Proof. Both sides of Eq. (2.12) for $\lambda = z - 1/z$ satisfy Eq. (2.8) and the same initial conditions. This proves (2.12) in view of the uniqueness of the solution of the initial value problem for this equation. Equation (2.13) can be obtained by substituting (2.12) into (2.11). □

We write

$$\mathsf{p}_N(z) = z^{2N}(z-b)(z-c) + (-1)^N(bz+1)(cz+1) \tag{2.14}$$

and also refer to $\mathsf{p}_N(z)$ as the characteristic polynomial. Clearly, studying the roots of $p_N(z)$ is essentially equivalent to studying the roots of $\mathsf{p}_N(z)$.

3. LOCATION OF THE ROOTS AS $N \to \infty$

Figure 1 shows the roots of the polynomial (2.14) for small and relatively large N. We see that all but four roots in Fig. 1(b) lie near the unit circle. This is a typical behavior of the roots as $N \to \infty$. Numerical experiment shows that for given b and c and for an arbitrarily small $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that all but at most four roots of the polynomial $\mathsf{p}_N(z)$ lie in an annulus of width 2ϵ for all $N > N_0$.

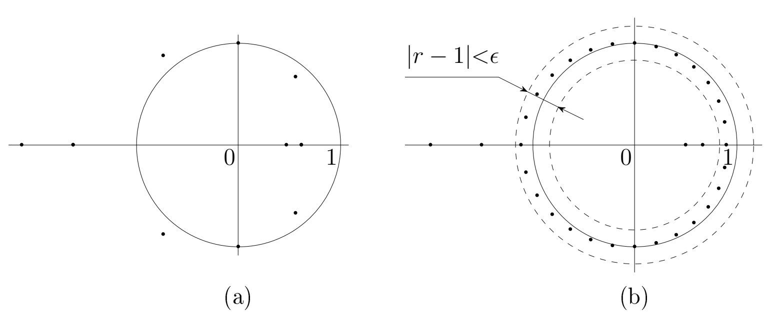

Fig. 1. The roots of the polynomial $\mathsf{p}_N(z)$ on the complex plane: (a) for $b = c = -2$ and $N = 5$; (b) for $b = -2$, $c = -3/2$, and $N = 15$. The dotted lines enclose the annulus containing roots close to the unit circle.

Next, we justify the numerical results analytically.

Lemma 2. *The polynomial* $\mathsf{p}(z)$ *is self-dual*; *that is,*

$$\mathsf{p}(z) = (-1)^N z^{2N+2}\mathsf{p}\left(-\frac{1}{z}\right).$$

Proof. The proof is by a straightforward verification. □

This lemma means that the roots of the polynomial $\mathsf{p}(z)$ form pairs of roots z and $-1/z$, which are dual to each other.

Remark 2. The paper uses the terms "dual roots" and "dual polynomials," which are not commonly recognized. The closely related terms "reciprocal roots" and "reciprocal polynomial" are not suitable because of the minus sign at $1/z$.

Lemma 3. *For $b = c$, the polynomial $\mathsf{p}_N(z)$ has the factorization*

$$\begin{aligned} \mathsf{p}_N(z) &= [z^N(z-b) - (1+bz)][z^N(z-b) + (1+bz)] \quad \text{for odd } N, \\ \mathsf{p}_N(z) &= [z^N(z-b) - i(1+bz)][z^N(z-b) + i(1+bz)] \quad \text{for even } N. \end{aligned} \tag{3.1}$$

Proof. The proof is by a straightforward verification. □

Theorem 1. *As $N \to \infty$, all roots of the polynomial $\mathsf{p}_N(z)$ except at most two pairs of dual roots tend to 1 in absolute value. The exceptional roots are described as follows.*

(i) *If $|b| > 1$ and $|c| > 1$, then these roots are distinct; one root tends to b, one root tends to c, and the third and fourth roots tend to the respective dual values. These four roots are real for $b \neq c$ as well as for $b = c$ and odd N; for $b = c$ and even N, they are strictly complex.*

(ii) *If $|b| > 1$ and $|c| \le 1$, then there exist two real roots, one of which tends to b and the other, to the dual value. A similar statement holds with interchanged $b \rightleftarrows c$.*

(iii) *If $|b| \le 1$ and $|c| \le 1$, then there exist no exceptional roots.*

Proof. Set $r = |z|$. Take b, c, and an $\epsilon_0 > 0$ such that the annulus $|r-1| \le \epsilon_0$ does not contain the points b and c and the dual points $-1/b$ and $-1/c$ provided that $b, c \neq \pm 1$. Obviously, any narrower annulus

$$|r-1| \le \epsilon, \quad \epsilon > 0, \tag{3.2}$$

where $\epsilon < \epsilon_0$ (see Fig. 1b), satisfies this property as well. We must show that for any $\epsilon < \epsilon_0$ there exists an N_0 such that, for all $N > N_0$, all but at most four roots of the polynomial $\mathsf{p}(z)$ lie in the annulus (3.2) and assertions (i)–(iii) hold. For convenience, we split the proof into parts.

1. In this part and part 2, we show that for any $\epsilon > 0$ and all sufficiently large N all but at most four roots of the polynomial $\mathsf{p}_N(z)$ lie in the annulus (3.2).

Set

$$f_N(z) = z^{2N}(z-b)(z-c), \quad g_N(z) = (-1)^N(1+bz)(1+cz). \tag{3.3}$$

On the boundary circle $r = 1 - \epsilon$ of the annulus (3.2), we have

$$\begin{aligned} |f_N(z)| &\le r^{2N}(r+|b|)(r+|c|) \to 0, \qquad N \to \infty; \\ |g_N(z)| &\ge |(1-|b|r)(1-|c|r)| = \delta > 0, \end{aligned} \tag{3.4}$$

where δ is independent of N. The estimates (3.4) are satisfied because b, c, and their dual points do not lie on the circle. Take an N_0 in such a way that the inequality

$$|f_N(z)| < |g_N(z)| \tag{3.5}$$

is satisfied on this circle for all $N \ge N_0$. This is possible by virtue of the estimates (3.4). Applying Rouché's theorem to the polynomials (3.3) and taking into account (3.5), we see that the number of roots of the polynomial $\mathsf{p}_N(z)$ in the disk $r < 1 - \epsilon$ coincides with that of the roots of the polynomial $g_N(z)$. The latter polynomial, and hence the polynomial $\mathsf{p}_N(z)$ as well, has two roots if $|b| > 1$ and $|c| > 1$, one root if $|b| > 1$ and $|c| \le 1$, and no roots if $|b| \le 1$ and $|c| \le 1$.

2. The circle $r = 1 + \epsilon$ can be considered in a similar way. On thus circle, we have

$$|f_N(z)| \ge r^{2N}|(r - |b|)(r - |c|)| \to \infty, \qquad N \to \infty,$$
$$|g_N(z)| \le (1 + |b|r)(1 + |c|r),$$

and therefore, there exists an N_0 such that

$$|g_N(z)| < |f_N(z)| \quad \text{for all } N \ge N_0.$$

By Rouché's theorem, the numbers of roots of the polynomials $\mathsf{p}_N(z)$ and $f_N(z)$ in the disk $r < 1 + \epsilon$ coincide. The latter polynomial, and hence $\mathsf{p}_N(z)$ as well, has $2N$ roots if $|b| > 1$ and $|c| > 1$, $2N + 1$ roots if $|b| > 1$ and $|c| \le 1$, and $2N + 2$ roots if $|b| \le 1$ and $|c| \le 1$.

3. It is convenient to refer to the roots that lie outside the annulus (3.2) for any $\epsilon > 0$ and all sufficiently large N as "exceptional roots." In this part of the proof, we show that the exceptional roots are separated for $|b| > 1$ and $|b| \ne |c|$.

If $|c| \le 1$, then we have only two exceptional roots. These roots are dual and not equal to $\pm i$, and hence they are separated. If $|c| > 1$, then we have four exceptional roots. To be definite, take $|b| < |c|$ and consider a circle whose radius satisfies the relation

$$|b| < r = |c| - \epsilon_1, \qquad \epsilon_1 > 0.$$

For sufficiently small ϵ_1, this circle separates b and c. By repeating the argument in part 2 of the proof for this circle, we see that the polynomial $\mathsf{p}_N(z)$ has exactly $2N - 1$ roots inside the circle for sufficiently large N. Since there are exactly two roots outside the circle $r = 1 - \epsilon$, we conclude that the circle separates these roots. The exceptional roots inside the circle $r = 1 - \epsilon$ are dual to the roots considered above and hence are separated as well.

4. In this part of the proof, we consider the case of $|b| > 1$ and $|b| \ne |c|$, prove that the exceptional roots are real in this case, and find their accumulation points as $N \to \infty$.

Assume that there exists a complex root. Then its complex conjugate is a root as well. These roots cannot be separated by a circle centered at zero, which contradicts part 3 of the proof. Therefore, these roots are real.

Let us show that if $|c| > 1$, then the roots tend to b, c, and their dual values. Consider the circle $r = 1 - \epsilon$ in part 1. We already know that as $N \to \infty$ the polynomial $\mathsf{p}_N(z)$ has exactly two roots in the disk $r < 1 - \epsilon$. An estimate similar to the first one in (3.4) shows that $f_N(z) \to 0$ for these roots. Consequently, $g_N(z) \to 0$ for these roots. It follows that the roots tend to $-1/b$ and $-1/c$ as $N \to \infty$. Accordingly, the dual roots tend to the dual values. By part 3, the limit values of the roots are separated. The case of $|c| \le 1$ can be considered in a similar way.

5. In this final part of the proof, we consider the case of $b = c$. Under this condition, $\mathsf{p}_N(z)$ factorizes as in (3.1). By reproducing the arguments in parts 1 and 2 for the factors, we conclude that for each factor there exist exactly two distinct exceptional roots for $|b| > 1$, and there exist no such roots for $|b| \le 1$. The exceptional roots of the two equations (3.1) are distinct, because the factors are dual to each other for odd N and complex conjugate for even N. Note that the roots cannot be multiple, because a multiple root would be a root of both factors, which can be verified to be impossible. Thus, the exceptional roots are real for odd N and complex conjugate for even N. $\square$

Remark 3. For $b + c = 0$, all roots of the polynomial $\mathsf{p}_N(z)$ lie on the unit circle. This follows from the fact that the polynomial $\mathsf{p}_N(z)$ is even for $b + c = 0$.

We conclude this section by describing the location of roots of the polynomial $p_N(\lambda)$ as $N \to \infty$.

Theorem 2. *As $N \to \infty$, almost all roots of the polynomial $p_N(\lambda)$ tend to the closed interval $2i \le \lambda \le 2i$. There exist at most two exceptional roots, which can be described as follows.*

(i) *If $|b| > 1$ and $|c| > 1$, then these roots are distinct*; *one of them tends to $b - 1/b$, and the other to $c - 1/c$. For $b \ne c$, as well as for $b = c$ and odd N, these roots are real*; *for $b = c$ and even N, they are strictly complex and complex conjugate.*

(ii) *If $|b| > 1$ and $|c| \le 1$, then there exists one real root tending to $b - 1/b$. A similar statement is true with b and c interchanged.*

(iii) *If $|b| \le 1$ and $|c| \le 1$, then there exist no such roots.*

4. SIMPLICITY OF ROOTS

In this section, we show that for arbitrarily chosen b and c (except for $b=-c=\pm 1$) and for all sufficiently large N the polynomial $\mathsf{p}_N(z)$ has no multiple roots.

Lemma 4. *The multiple roots of the polynomial $\mathsf{p}_N(z)$, if any, are also roots of the polynomial*

$$\mathsf{m}_N(z)=2N(z-b)(z-c)(1+bz)(1+cz)+z(1+bc)[(b+c)z^2+2(1-bc)z-b-c]. \tag{4.1}$$

Proof. The multiple roots of $\mathsf{p}_N(z)$ are also roots of the polynomial

$$\dot{\mathsf{p}}_N(z)=2Nz^{2N-1}(z-b)(z-c)+z^{2N}(2z-b-c)+(-1)^N(2bcz+b+c).$$

Assuming that b and c are not roots of $\mathsf{p}_N(z)$, we eliminate z^{2N} from the system of equations $\mathsf{p}_N(z)=0$, $\dot{\mathsf{p}}_N(z)=0$ to obtain (4.1). If, say, b is a root of $\mathsf{p}_N(z)$, then a verification shows that the condition $\mathsf{p}_N(b)=0$ implies $bc=-1$; i.e., (4.1) is also satisfied in this case. □

Remark 4. The roots of the polynomial $\mathsf{m}_N(z)$ can be found explicitly, without resorting to known methods for solving algebraic equations of the fourth degree. This happens because the polynomial $\mathsf{m}_N(z)$ is self-dual,

$$z^4\mathsf{m}_N\left(-\frac{1}{z}\right)=\mathsf{m}_N(z),$$

and hence can be expressed via the natural spectral variable,

$$m(\lambda)=2N(b\lambda+1-b^2)(c\lambda+1-c^2)+(1+bc)[(b+c)\lambda+2(1-bc)].$$

Lemma 5. *If $bc=-1$, then the polynomial $\mathsf{p}_N(z)$ has multiple roots only for $|b|=1$ and even N. These are the roots $z=\pm 1$ of multiplicity two.*

Proof. For $bc=-1$, we have

$$\mathsf{p}_N(z)=(z-b)(z+1/b)[z^{2N}-(-1)^N],$$

whence the assertion of the lemma follows. □

Lemma 6. *If $b\neq c$ and $bc\neq -1$, then the roots of the polynomial $\mathsf{m}_N(z)$ have the following asymptotics as $N\to\infty$:*

$$\begin{aligned} z&=b-\frac{b}{2N}+\cdots, & z&=c-\frac{c}{2N}+\cdots,\\ z&=-\frac{1}{b}-\frac{1}{2bN}+\cdots, & z&=-\frac{1}{c}-\frac{1}{2cN}+\cdots, \end{aligned} \tag{4.2}$$

where the dots stand for terms with higher powers of $1/N$.

For $b=c$, the corresponding asymptotics have the form

$$z=b,\quad z=-\frac{1}{b},\quad z=b-\frac{b}{N}+\cdots,\quad z=-\frac{1}{b}-\frac{1}{bN}+\cdots. \tag{4.3}$$

Proof. The expansions (4.2) and (4.3) can be verified by substituting them into (4.1) and then calculating the asymptotics as $N\to\infty$. □

Theorem 3. *For any b and c except for $b=-c=\pm 1$, there exists an $N_0\in\mathbb{N}$ such that the polynomial $\mathsf{p}_N(z)$ has no multiple roots for any $N>N_0$. In the case of $b=-c=\pm 1$, multiple roots exist only for even N; these are double roots ± 1.*

Proof. Let us show that the estimates (4.2), (4.3) are inconsistent with the behavior of the roots of the polynomial $\mathsf{p}_N(z)$ as $N \to \infty$. Consider, say, the first estimate in (4.2). Substituting it into (2.14) and calculating the asymptotics, we obtain

$$\mathsf{p}_N(z) = b^{2N}\left[-\frac{b(b-c)}{2eN} + \cdots\right], \quad |b| > 1,$$
$$\mathsf{p}_N(z) = (-1)^N(1+b^2)(1+bc) + \cdots, \quad |b| \le 1;$$

i.e., $\mathsf{p}_N(z) \neq 0$. The remaining estimates can be considered in a similar way. The case of $b = -c = \pm 1$ is considered in Lemma 5. □

In terms of the natural spectral variable, Theorem 3 can be stated as follows.

Theorem 4. *For arbitrary b and c except for $b = -c = \pm 1$ and for all sufficiently large N, the polynomial $p_N(\lambda)$ has no multiple roots. In the case of $b = -c = \pm 1$, there exists a multiple root only for even N; it is unique and is equal to zero.*

5. STABILITY OF THE CHARACTERISTIC POLYNOMIAL

A polynomial is said to be *stable* if all of its roots lie in the open left complex half-plane. In this section, we study the stability of the polynomial $p_N(\lambda)$ as a function of the parameters b and c. For the stability criterion we take the following statement. This is a slightly modified version of [20, Theorem XVII] or the corresponding statement in [21, Theorem 9.11].

Theorem 5. *Let p be a real polynomial, and let $p = p^I + p^{II}$ be its decomposition into even and odd parts. The polynomial p is stable if and only if the leading coefficients of the polynomials p^I and p^{II} have the same sign and their roots are imaginary and simple and satisfy the interlacing property.*

According to this criterion, we decompose the polynomial $p_N(\lambda)$ into an even part (a polynomial with even powers of λ) and an odd part (a polynomial with odd powers of λ) and write

$$p_N(\lambda) = p_N^I(\lambda) + p_N^{II}(\lambda), \tag{5.1}$$
$$p_N^I(\lambda) = P_N(\lambda) + bcP_{N-2}(\lambda), \qquad p_N^{II}(\lambda) = -(b+c)P_{N-1}(\lambda). \tag{5.2}$$

The polynomials $p_N^I(\lambda)$ and $p_N^{II}(\lambda)$ inherit a number of properties of the polynomials $P_N(\lambda)$. In particular, they satisfy the identities

$$p_l^I(-\lambda) = (-1)^l p_l^I(\lambda), \qquad p_l^{II}(-\lambda) = (-1)^{l+1} p_l^{II}(\lambda). \tag{5.3}$$

The stability analysis is rather cumbersome. Therefore, we split it into several small parts stated as lemmas.

Lemma 7. *For $b + c \ge 0$, as well as for $bc < -1$ and even N, the polynomial $p_N(\lambda)$ is unstable. For $bc = -1$, the polynomial $p_N(\lambda)$ is unstable for any N.*

Proof. If $b + c > 0$, then, according to (5.1) and (5.2), the leading and the next coefficients of the polynomial $p_N(\lambda)$ have opposite signs, and therefore, the Stodola condition is not satisfied for the polynomial [22]. In a similar way, it follows from (5.2) and (2.12) that for even N the constant term of the polynomial $p_N^I(\lambda)$ has the form $1 + bc$, and therefore, for $bc < -1$ the Stodola condition is not satisfied for $p_N(\lambda)$ either.

If $b + c = 0$, then $p_N^{II} = 0$, and consequently, the stability criterion in Theorem 5 is violated. For $bc = -1$, the roots of the polynomial $p_N(\lambda) = 0$ can be calculated explicitly. □

Lemma 8. *For $b + c \neq 0$, the roots of the polynomial $p_N^{II}(\lambda)$, $N \ge 2$, have the form*

$$\lambda_k = -2i\cos\frac{\pi k}{N}, \qquad k = 1, \ldots, N-1, \tag{5.4}$$

and therefore, they are imaginary and simple and lie on the interval $[\lambda_1, \lambda_{N-1}]$.

Proof. According to (2.12), the roots of the polynomial $p_N^{II}(\lambda)$ are solutions of the equation

$$z^{2N} - (-1)^N = 0$$

for z except for the zeros $z = \pm i$. We represent the roots in the form

$$z_k = e^{-\frac{i\pi}{2}+\frac{i\pi k}{N}}, \quad k = 1, \dots, N-1. \tag{5.5}$$

Passing to the variable λ, we obtain (5.4). □

Lemma 9. *The polynomial $p_N^I(\lambda)$ takes the following values on the roots of the polynomial $p_N^{II}(\lambda)$, $N \geq 2$, ordered according to* (5.4):

$$p_N^I(\lambda_k) = (-i)^N(1+bc)(-1)^k, \quad k = 1, \dots, N-1. \tag{5.6}$$

Therefore, between any two adjacent roots of the polynomial $p_N^{II}(\lambda)$ there exists at least one root of the polynomial $p_N^I(\lambda)$.

Proof. To obtain (5.6), we need to express $p_N^I(\lambda)$ via z and substitute (5.5) into the resulting expression. The calculations are simplified if we note that $P_N(\lambda_k) = P_{N-2}(\lambda_k)$. □

Lemma 10. *For $bc < -1$ and odd $N \geq 2$, the polynomial $p_N(\lambda)$ is unstable.*

Proof. For odd N, the coefficient of λ in the polynomial $p_N^I(\lambda)$ is calculated as follows:

$$\begin{aligned}
\frac{dp_N^I(\lambda)}{d\lambda}\Big|_{\lambda=0} &= \Big[\Big(\frac{d\lambda}{dz}\Big)^{-1}\frac{dp_N^I(\lambda)}{dz}\Big]_{z=\pm 1} \\
&= \frac{1}{2}\frac{dp_N^I(\lambda)}{dz}\Big|_{z=\pm 1} \\
&= \frac{1}{2}\frac{d}{dz}\Big[\frac{z^{2N+2}+(-1)^N}{z^N(1+z^2)} + bc\frac{z^{2N-2}+(-1)^N}{z^{N-2}(1+z^2)}\Big]_{z=\pm 1} \\
&= \frac{1}{2}[N+1+bc(N-1)].
\end{aligned}$$

We see that for

$$bc < -\frac{N+1}{N-1}$$

this coefficient is negative, and therefore, the Stodola condition is violated for the polynomial $p_N(\lambda)$; i.e., $p_N(\lambda)$ is unstable.

Assume that

$$-\frac{N+1}{N-1} < bc < -1.$$

In this case, the coefficient in question is positive, and

$$p_N^I(\pm i0) = \frac{\pm i0}{2}[N+1+bc(N-1)]. \tag{5.7}$$

On the other hand, in accordance with (5.6), on the roots $\lambda_{(N\pm 1)/2}$ of $p_N^{II}(\lambda)$ closest to zero, the polynomial $p_N^I(\lambda)$ takes the values

$$p_N^I\big(\lambda_{(N\pm 1)/2}\big) = \pm i(1+bc). \tag{5.8}$$

Comparing the values (5.7) and (5.8), we find that the polynomial $p_N^I(\lambda)$ changes sign three times on the interval

$$\lambda_{(N-1)/2} < \lambda < \lambda_{(N+1)/2};$$

i.e., the roots of the polynomials $p_N^I(\lambda)$ and $p_N^{II}(\lambda)$ do not satisfy the interlacing property. □

The subsequent stability analysis is based on the technique of finite orthogonal polynomials.

Lemma 11. *Let $y_l(\lambda)$ be a solution of the recurrence relation* (2.8). *The following identities hold*:

$$\begin{vmatrix} y_{l-1}(\lambda) & y_{l-1}(\mu) \\ y_l(\lambda) & y_l(\mu) \end{vmatrix} = (-1)^l \begin{vmatrix} y_{-1}(\lambda) & y_{-1}(\mu) \\ y_0(\lambda) & y_0(\mu) \end{vmatrix} + (\lambda-\mu)\sum_{l'=0}^{l-1}(-1)^{l-l'} y_{l'}(\lambda) y_{l'}(\mu), \tag{5.9}$$

$$\begin{vmatrix} \dot y_{l-1}(\lambda) & y_{l-1}(\lambda) \\ \dot y_l(\lambda) & y_l(\lambda) \end{vmatrix} = (-1)^l \begin{vmatrix} \dot y_{-1}(\lambda) & y_{-1}(\lambda) \\ \dot y_0(\lambda) & y_0(\lambda) \end{vmatrix} + \sum_{l'=0}^{l-1}(-1)^{l'-l}[y_{l'}(\lambda)]^2. \tag{5.10}$$

Proof. Let us write (2.8) for two arguments,

$$\lambda y_l(\lambda) = y_{l+1}(\lambda) - y_{l-1}(\lambda),$$
$$\mu y_l(\mu) = y_{l+1}(\mu) - y_{l-1}(\mu),$$

and subtract the second equation multiplied by $y_l(\mu)$ from the first equation multiplied by $y_l(\lambda)$. We obtain the recurrence relation

$$(\lambda-\mu) y_l(\lambda) y_l(\mu) = \begin{vmatrix} y_{l+1}(\lambda) & y_{l+1}(\mu) \\ y_l(\lambda) & y_l(\mu) \end{vmatrix} + \begin{vmatrix} y_l(\lambda) & y_l(\mu) \\ y_{l-1}(\lambda) & y_{l-1}(\mu) \end{vmatrix},$$

whose solution is given by (5.9). Identity (5.10) follows from (5.9) by l'Hôpital's rule. ☐

Lemma 12. *For $bc > -1$, the roots of the polynomial $p^I_N(\lambda)$ are imaginary and simple.*

Proof. Assume that the root λ of the polynomial $p^I_N(\lambda)$ has a nonzero real part; that is, $\lambda + \bar\lambda \neq 0$. Applying formula (5.9) to this polynomial with $\mu = -\bar\lambda$, we obtain

$$\begin{vmatrix} p^I_{N-1}(\lambda) & p^I_{N-1}(-\bar\lambda) \\ p^I_N(\lambda) & p^I_N(-\bar\lambda) \end{vmatrix} = (-1)^N \begin{vmatrix} -bc\lambda & bc\bar\lambda \\ bc+1 & bc+1 \end{vmatrix} + (\lambda+\bar\lambda)\sum_{l=0}^{N-1}(-1)^{N-l} p^I_l(\lambda) p^I_l(-\bar\lambda)$$
$$= (-1)^N(\lambda+\bar\lambda)\Big[1 + bc + \sum_{l=1}^{N-1} p^I_l(\lambda) p^I_l(\bar\lambda)\Big],$$

where, in the last equality, we have used (5.3). The left-hand side of this chain of equalities, when taking into account (5.3), is zero, and the right-hand side is nonzero for $bc > -1$. We have arrived at a contradiction. Consequently, all roots of the polynomial $p^I_N(\lambda)$ are imaginary for $bc > -1$.

To prove that the roots are simple, one follows a similar scheme using identity (5.10) instead of (5.9). ☐

Lemma 13. *For $bc > -1$ and $b + c \neq 0$, the roots of the polynomials $p^I_N(\lambda)$ and $p^{II}_N(\lambda)$ satisfy the interlacing property.*

Proof. According to (5.2) and (5.10),

$$\dot p^I_N(\lambda) p^{II}_N(\lambda) - p^I_N(\lambda)\dot p^{II}_N(\lambda) = (b+c)\begin{vmatrix} \dot P_{N-1}(\lambda) & P_{N-1}(\lambda) \\ \dot P_N(\lambda) & P_N(\lambda) \end{vmatrix} - bc(b+c)\begin{vmatrix} \dot P_{N-2}(\lambda) & P_{N-2}(\lambda) \\ \dot P_{N-1}(\lambda) & P_{N-1}(\lambda) \end{vmatrix}$$
$$= (-1)^N(b+c)\left[\sum_{l=1}^{N-1}|P_l(\lambda)|^2 + bc\sum_{l=1}^{N-2}|P_l(\lambda)|^2\right],$$

and therefore, for $bc > -1$ one has

$$(-1)^N(b+c)^{-1}\big[\dot p^I_N(\lambda) p^{II}_N(\lambda) - p^I_N(\lambda)\dot p^{II}_N(\lambda)\big] > 0. \tag{5.11}$$

Let λ_1 and λ_2 be two adjacent roots of the polynomial $p^I_N(\lambda)$ with respect to the natural order on the imaginary axis. Since $p^I_N(\lambda)$ has only simple roots, it follows that $\dot p^I_N(\lambda_1)$ and $\dot p^I_N(\lambda_2)$ have opposite signs. Then it follows from (5.11) that $p_N^{II}(\lambda_1)$ and $p_N^{II}(\lambda_2)$ have opposite signs as well. This proves that between two roots of the polynomial $p_N^{I}(\lambda)$ there lies a root of the polynomial $p_N^{II}(\lambda)$. The proof that between two adjacent roots of the polynomial $p_N^{II}(\lambda)$ there lies a root of the polynomial $p_N^{I}(\lambda)$ is similar. □

Theorem 6. *The polynomial $p_N(\lambda)$, $N \ge 2$, is stable if and only if $bc > -1$ and $b + c < 0$.*

Proof. This follows from Theorem 5 and Lemmas 7–13. □

6. ORTHOGONALITY AND COMPLETENESS OF EIGENVECTORS

For an arbitrary pair of vectors

$$u = \begin{pmatrix} u_0 \\ \dots \\ u_{N-1} \end{pmatrix}, \qquad v = \begin{pmatrix} v_0 \\ \dots \\ v_{N-1} \end{pmatrix},$$

we write

$$\langle u, v\rangle = \sum_{l=0}^{N-1} (-1)^{N-l} u_l v_l. \tag{6.1}$$

In particular,

$$\langle \gamma_k, \gamma_{k'}\rangle = \sum_{l=0}^{N-1} (-1)^{N-l} p_l(\lambda_k) p_l(\lambda_{k'}), \quad k = 0, \dots, N-1,$$

for the eigenvectors of the matrix A, where λ_k and $\lambda_{k'}$ are roots of the polynomial $p_N(\lambda)$.

Lemma 14. *The following identities hold*:

$$\begin{vmatrix} p_{N-1}(\lambda) & p_{N-1}(\mu) \\ p_N(\lambda) & p_N(\mu) \end{vmatrix} = (\lambda - \mu) \sum_{l'=0}^{N-1} (-1)^{N-l'} p_{l'}(\lambda) p_{l'}(\mu), \tag{6.2}$$

$$\begin{vmatrix} \dot p_{N-1}(\lambda) & p_{N-1}(\lambda) \\ \dot p_N(\lambda) & p_N(\lambda) \end{vmatrix} = \sum_{l'=0}^{N-1} (-1)^{N-l'} [p_{l'}(\lambda)]^2. \tag{6.3}$$

Proof. Identities (6.2) and (6.3) follow from the definition of the polynomials $p_l(\lambda)$ and identities (5.9) and (5.10) for $l = N$. □

Theorem 7. *For arbitrarily chosen b and c and for all sufficiently large N, except for the case in which $b = -c = \pm 1$ and N is even, the eigenvectors form a complete set and are orthogonal with respect to the form* (6.1):

$$\langle \gamma_k, \gamma_{k'}\rangle = \langle \gamma_k, \gamma_k\rangle \delta_{k,k'}, \qquad k, k' = 0, \dots, N-1, \tag{6.4}$$

$$\langle \gamma_k, \gamma_k\rangle \ne 0, \qquad k = 0, \dots, N-1. \tag{6.5}$$

Proof. By Theorem 4, all eigenvalues of A are simple, and therefore, the eigenvectors form a complete set. Then (6.4) follows from (6.2) for $\lambda = \lambda_k$ and $\mu = \lambda_{k'}$.

Let us verify (6.5). Assume the contrary: $\langle \gamma_k, \gamma_k\rangle = 0$ for some simple root λ_k. Then

$$p_N(\lambda_k) = 0, \qquad \dot p_N(\lambda_k) \ne 0,$$

and it follows from (6.3) that

$$p_{N-1}(\lambda_k) = 0.$$

According to (2.7), this means that

$$y_{N-1}(\lambda_k) = y_N(\lambda_k) = 0.$$

From this and from (2.4), we see that

$$y_l(\lambda_k) = 0, \qquad l = -1, \dots, N.$$

We have arrived at a contradiction. □

7. APPENDIX

The motion of oscillators in a homogeneous harmonic chain is described by the system of equations

$$\ddot{q}_l - q_{l+1} + 2q_l - q_{l-1} = 0, \qquad l = 0, \dots, L-1, \tag{7.1}$$

where q_l is the displacement of the lth oscillator in the chain from the equilibrium position.

To make the dynamics unambiguous, one specifies boundary conditions in addition to the initial conditions; i.e., the displacements q_{-1} and q_L are specified. A distinction is made between conservative and nonconservative boundary conditions. The first ones have the following representatives:

$\dot{q}_{-1} = \dot{q}_L = 0$, a chain with fixed ends.

$q_{-1} = q_0$ and $q_{L-1} = q_L$, a chain with free ends.

$q_{-1} = q_0$ and $\dot{q}_L = 0$, a chain with free left end and fixed right end.

Any of these conditions turns (7.1) into a linear conservative system. Owing to the conservation of total energy, the oscillations in such a system do not decay or grow unboundedly. From the viewpoint of spectral theory, this means that the eigenfrequencies of the system are real and the eigenoscillations form a complete system.

The simplest boundary conditions under which the energy dissipation and/or generation can occur are the conditions

$$\begin{gathered} b\dot{q}_0 + q_0 - q_{-1} = 0, \\ c\dot{q}_{L-1} - q_L + q_{L-1} = 0 \end{gathered} \tag{7.2}$$

or

$$\begin{gathered} b\dot{q}_0 + q_0 - q_{-1} = 0, \\ \dot{q}_L - c(q_L - q_{L-1}) = 0, \end{gathered} \tag{7.3}$$

where $b, c \in \mathbb{R}$. Under these conditions, the eigenfrequencies of the system can take complex values.

Schrödinger variables. The transition from Eqs. (7.1)–(7.3) to the dynamical system (2.1), (2.2) is carried out by means of the Schrödinger variables, which we introduce by the expressions

$$x_{2l} = \dot{q}_l, \qquad x_{2l+1} = q_{l+1} - q_l, \qquad l = 0, \dots, L-1. \tag{7.4}$$

Proposition 2. *In the Schrödinger variables, problems* (7.1), (7.2) *and* (7.1), (7.3) *are combined into the single boundary value problem*

$$\dot{x}_l = x_{l+1} - x_{l-1}, \quad l = 0, \dots, N-1, \tag{7.5}$$

$$\begin{gathered} x_{-1} + bx_0 = 0, \\ cx_{N-1} - x_N = 0, \end{gathered} \tag{7.6}$$

where $N = 2L - 1$ for problem (7.1), (7.2) *and $N = 2L$ for problem* (7.1), (7.3).

Proof. The equivalence of (7.1) and (7.5) in view of (7.4) is justified by the following chains of equalities:

$$\dot{x}_{2l} = \ddot{q}_l = q_{l+1} - 2q_l + q_{l-1} = x_{2l+1} - x_{2l-1},$$
$$\dot{x}_{2l+1} = \dot{q}_{l+1} - \dot{q}_l = x_{2l+2} - x_{2l}.$$

The equivalence of boundary conditions (7.2) and (7.3) to condition (7.6) follows by a straightforward verification using (7.4) and the relationship between the parameters L and N. Namely,

$$b\dot{q}_0 + q_0 - q_{-1} = 0 \iff bx_0 + x_{-1} = 0,$$
$$c\dot{q}_{L-1} - q_L + q_{L-1} = 0 \iff cx_{2L-2} - x_{2L-1} = 0 \iff cx_{N-1} - x_N = 0,$$
$$\dot{q}_L - c(q_L - q_{L-1}) = 0 \iff x_{2L} - cx_{2L-1} = 0 \iff x_N - cx_{N-1} = 0,$$

where N is odd in the second chain of equivalences and even in the last one. □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. A. Maradudin, E. Montroll, and G. Weiss, *Theory of Lattice Dynamics in the Harmonic Approximation* (Academic Press, New York–London, 1963).
2. I. Newton, *The Principia: The Authoritative Translation. Mathematical Principles of Natural Philosophy* (University of California Press, Oakland, CA, 2016).
3. *Thermal Transport in Low Dimensions*, in *Lecture Notes in Physics*, Ed. by S. Lepri (Springer International Publishing, New York, 2016), Vol. 921.
4. G. A. Weiderpass, G. M. Monteiro, and A. O. Caldeir, "Exact solution for the heat conductance in harmonic chains," Phys. Rev. B **102** (12), 125401(8) (2020).
5. A. Dhar, "Heat transport in low-dimensional systems," Advances in Physics **57** (5), 457–537 (2008).
6. F. P. Gantmacher, *The Theory of Matrices,* Vol. 1 (AMS Chelsea Publishing, Providence, RI, 1998).
7. F. P. Gantmacher and M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems* (AMS Chelsea Publishing, Providence, RI, 2002).
8. G. M. L. Gladwell, "Matrix inverse eigenvalue problems," in *Dynamical Inverse Problems: Theory and Application*, *CISM Courses and Lectures*, Ed. by G. M. L. Gladwell and A. Morassi (Springer Wien, New York, 2011), Vol. 529.
9. M. T. Chu and G. H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications* (Oxford University Press, Oxford, 2005).
10. G. Sh. Guseinov, "On an inverse problem for two spectra of finite Jacobi matrices," Appl. Math. Comput. **218**, 7573–7589 (2012).
11. A. I. Gudimenko, "Inverse spectral problem for an antisymmetric tridiagonal matrix," Sib. Èlektron. Mat. Izv. **20** (2), 1026–1036 (2023).
12. F. V. Atkinson, *Discrete and Continuous Boundary Problems* (Academic Press, New York, 1964).
13. N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, Edinburgh, 1965).
14. Yu. Arlinskii and E. Tsekanovskii, "Non-self-adjoint Jacobi matrices with a rank-one imaginary part," J. Funct. Anal. **241**, 383–438 (2006).
15. E. Ergun, "On the inverse problem for finite dissipative Jacobi matrices with a rank-one imaginary part," Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (2), 1273–1288 (2019).
16. E. Schrödinger, "Zur Dynamik elastisch gekoppelter Punktsysteme," Annalen der Physik **44**, 916–934 (1914).
17. E. Takizawa and K. Kobayasi, "Heat flow in a system of coupled harmonic oscillators," Chinese J. Phys. **1** (2), 59–73 (1963).

18. A. I. Gudimenko and A. V. Lihosherstov, “On the phenomenon of low-frequency, large-amplitude oscillations in a high-dimensional linear dynamical system,” Rus. J. Nonlin. Dyn. **20** (2), 259–276 (2024).
19. A. A. Samarskii, *The Theory of Difference Schemes* (Marcel Dekker, Inc., New York, 2001).
20. M. G. Krein and M. A. Naimark, “The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations,” Linear and Multilinear Algebra **10**, 265–308 (1981).
21. P. A. Fuhrmann, *A Polynomial Approach to Linear Algebra* (Springer-Verlag, New York–Dordrecht–Heidelberg–London, 2012).
22. M. M. Postnikov, *Stable Polynomials* (Nauka, Moscow, 1981) [in Russian].